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Ph. D. Thesis

Classical Histories In Hamiltonian Systems

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Part I:

Introduction

Chapter I-1. Overview Of The Thesis.

● The Aim Of The Thesis: The Hamiltonian formulation of general relativity was developed originally as a preliminary step to a quantum theory of gravity. Time is separated from the remaining spacetime coordinates and is treated as a privileged parameter, in accordance with the standard algorithm of quantization. The incompatibility between the treatment of time in the classical and in the quantum theory results in the so-called problem of time in canonical quantum gravity[1,2]. Several attempts have been made to devise alternative algorithms of quantization which may accommodate the covariance of the classical theory from the outset.

One of the most prominent of these attempts is based on the notion of continuous histories[3,4,5,6] in the context of the consistent histories approach to quantum theory[7,8,9,10]. A *history* is defined as a sequence of *time-ordered* propositions about the properties of the physical system. It is precisely this intrinsic temporality of histories that may provide a solution to the problem of time in quantum gravity. Such a solution is the ultimate aim of the histories program. The analogue of continuous histories in a classical theory is the main theme of this thesis.

By the term *classical histories* it is implied that the canonical fields, as well as the symplectic structure of the theory, depend on the full foliation of spacetime into hypersurfaces, rather than on the embedding at a single instant of time. For example, the history Poisson bracket between a scalar field and its conjugate momentum is defined by

$\{\phi(x, t), \pi(x', t')\} := \delta(x, x')\delta(t, t')$. The aim of the thesis is to illustrate that, even at the classical level, the advantages of a history theory over the standard canonical approach are significant, especially when it comes to discussing spacetime issues. This fact strongly suggests a history canonical approach when considering quantum gravity.

- The Context Of The Thesis: The need for using classical histories came from an attempt to deal with an apparently unrelated issue; namely, the lack of a genuine Lie algebra in canonical general relativity. More precisely, the Dirac algebra[11] is the algebra according to which the Hamiltonian and momentum constraints of canonical relativity close under the Poisson bracket operations. However, the induced metric on the three-dimensional hypersurface appears explicitly in the Poisson bracket between the Hamiltonian constraints defined at two spatially distinct points. This means that the Dirac algebra is not a genuine Lie algebra; a fact that creates many serious difficulties in canonical quantum gravity, especially in group-oriented approaches[1,2,12].

The coupling of gravity to dust helped Brown and Kuchař[13] to parameterized discover simple quadratic combinations of the gravitational Hamiltonian and momentum constraints whose Poisson brackets vanish strongly. If these combinations replace the Hamiltonian constraint to form an equivalent system of constraints for *vacuum* general relativity, a genuine Lie algebra is created. It is natural to ask whether the coupling of gravity to other sources yields alternative combinations of the gravitational constraints whose Poisson brackets also vanish strongly. Kuchař and Romano[14] illustrated how this can be done by coupling gravity to a massless scalar field. Brown and Marolf[15] produced other combinations by coupling gravity to fluids, and Markopoulou[16] found an equation satisfied

by all such combinations by treating the problem algebraically.

The physical relevance of this equation is not clear. However, an insight into its origin can be gained if it is shown to be related to phenomenologically physical systems like the ones discussed by Kuchař *et al.* Indeed, it is shown (here, and in ref. 17) that all such combinations can be derived from a generalized action functional that is coupled to gravity. The action functional depends on a scalar field and is by two arbitrary functions of a Lagrange multiplier. After the elimination of the multiplier from the action, the resulting theory is interpreted as a theory of gravity coupled to scalar fields with nonlinear self-interactions.

The momentum conjugate to the scalar field can be solved solely in terms of the gravitational Hamiltonian and momentum generators. This leads to scalar densitized combinations of the gravitational generators that are parametrized by an arbitrary function of one variable. In the case of *vacuum* gravity, these combinations provide exactly the general solution of the equation constructed algebraically by Markopoulou.

The coincidence of the purely algebraic result with the result arising from the action principle is indeed remarkable. It suggests that the role of couplings in general relativity ought to be investigated further. However, returning to the case of vacuum gravity, an important problem arises concerning the usefulness of these self-commuting combinations. Namely, it is shown (here) that most of the combinations are ill-defined on the constraint surface of vacuum gravity, while the remaining (well-defined) ones lead to a trivial dynamical evolution if used to replace the Hamiltonian constraint.

The question arises whether alternative gravitational constraints for vacuum gravity can

be constructed. These should not only satisfy the new Lie algebra but also generate the appropriate dynamical evolution. In order to investigate this possibility, a further insight into the origin of the new Lie algebra is required. Indeed, significant progress is achieved by comparing the new algebra with the Dirac one (here, and in ref. 18). The geometric interpretation of the latter is known and a method for finding its physically relevant representations is also available.

The underlying geometry of the Dirac algebra was first recognized by Teitelboim[19], while an algorithm for constructing its representations was developed by Hojman, Kuchař and Teitelboim in their derivation of geometrodynamics from first principles[20]. The attempt to apply their algorithm unambiguously to the case of the new algebra results in a re-examination of the standard canonical formalism and in the introduction of classical histories. This is precisely how the notion of classical histories arises in the thesis.

If the phase space is defined over the space of classical histories, it is shown (here, and in ref. 18) that the precise relation between the individual postulates used by Kuchař *et al* is clarified. The original set of postulates can then be replaced by an *evolution* postulate which is related directly to the spacetime picture. To be precise, the original aim of Kuchař *et al* was to use postulates that depend exclusively on the hyper-surface. The assumption of a surrounding spacetime appears only implicitly in their arguments, and this is the reason why the connection of the postulates is not clear.

It is only after choosing to make this assumption explicit that the precise relationship of the postulates is revealed. If this choice is not taken, nothing can be gained or lost through the introduction of histories. What is remarkable, however, is that in the standard formalism

there is *no* second choice, and the postulates of Kuchař *et al* are then the best one can hope for. This happens precisely because of the use of equal-time Poisson brackets in the standard approach. It is only in the history formalism that the assumption of a surrounding spacetime can be made explicit, thus leading to the understanding of the postulates.

This observation, alone, illustrates the genuine advantages of the canonical history approach over its standard counterpart and speaks in its when approaching quantum gravity.

In addition, the use of a history phase space results in certain corrections of the results of Kuchař *et al*. These lead to the discovery of additional representations of the evolution postulate for general relativity. The meaning of these new representations is only partially clarified in the thesis.

Finally, by applying the history formalism to the original problem of the new algebra, its geometric interpretation is found. The representations of the algebra that generate the correct dynamical evolution are also constructed. This is achieved by decomposing the Lagrangian with respect to an appropriate choice of foliation. The Hamiltonian and momentum constraints that arise through this decomposition are then the required representations of the algebra. Since the choice of foliation does not affect the physical content of the theory, it follows that these representations generate the correct dynamical evolution. This procedure works for an arbitrary algebra and for any field theory that is parametrized. For general relativity a more elaborate scheme needs to be devised; this project is under preparation[21].

- The Structure Of The Thesis: A brief review of the standard canonical formulation of general relativity is presented in the remaining part of the introduction. The problem of

time as well as the main approaches to quantization are discussed. Particular emphasis is placed on the internal-time approach and on the related Gaussian-time formulation, the knowledge of which is needed for the chapters that follow. Part II begins with the presentation of the genuine Lie algebra discovered by Brown and Kuchař in the context of the Gaussian-time approach. Certain representations of this algebra are combinations of the super-Hamiltonian and super-momentum constraints of canonical general relativity. The connection of these combinations with a generalized scalar field action functional that is coupled to gravity is presented.

In the case of pure gravity, these representations are ill-defined and an alternative interpretation is needed. This need leads to the use of classical histories which are discussed in two parts. In part III the phase space over the space of histories is defined while an algorithm for interpreting the algebra is explained. It modifies and completes the algorithm devised by Kuchař *et al*[20] for deriving representations of the canonical generators from first principles. In the new algorithm, the principles of Kuchař *et al* are replaced by the evolution postulate.

In part IV, the evolution postulate is used to find the most general canonical representations of (i) vacuum general relativity and (ii) a scalar field theory on a given metric background. These two theories act as examples of a constrained and an unconstrained system, respectively. In the former case, new canonical representations arise. In part V, the geometric interpretation of the new Lie algebra is presented. The revised algorithm is used in finding representations of the algebra for a parametrized scalar field theory. The current results, their extent and their limitations are discussed in part VI. Some future

directions, both in the classical and in the quantum domain, are pointed out.

Chapter I-2. Hamiltonian General Relativity.

- The theory of general relativity was written in Hamiltonian form initially by Dirac[11], Arnowitt, Deser and Misner[22]. A feature of the original formulation was the selection of a specific coordinate system on the spacetime manifold but, later, a global geometric approach was developed by Kuchař[23]. The following summary is based on an article by Isham[1].

- The Required Assumptions For Spacetime: The starting point is a 3-dimensional man-

ifold Σ that represents physical space. It is assumed to be compact. Without this assumption the resulting theory has to be augmented by surface terms. The topology of the spacetime manifold \mathcal{M} is assumed to be such that it can be foliated by an one-parameter family of embeddings,

$$\mathcal{X}_t : \Sigma \rightarrow \mathcal{M}, \quad (1)$$

$t \in R$, of Σ in \mathcal{M} . This implies that the spacetime manifold is limited topologically to be diffeomorphic to $\Sigma \times R$, since the map

$$\mathcal{X} : \Sigma \times R \rightarrow \mathcal{M}, \quad (2)$$

defined by $(x, t) \mapsto \mathcal{X}(x, t) := \mathcal{X}_t(x)$, $x \in \Sigma$, is a diffeomorphism of $\Sigma \times R$ with \mathcal{M} .

For each $x \in \Sigma$ the map

$$\mathcal{X}_x : R \rightarrow \mathcal{M}, \quad (3)$$

defined by $t \mapsto \mathcal{X}(x, t)$, is a curve in \mathcal{M} and therefore has an one-parameter family of tangent vectors in \mathcal{M} . This is known as the deformation vector field of the foliation and is defined by

$$\dot{\mathcal{X}}(x, t) := \dot{\mathcal{X}}_x(t). \quad (4)$$

For a particular choice of foliation there is a unique deformation vector field.

In general, if $\mathcal{X}_t : \Sigma \rightarrow \mathcal{M}$ is an embedding, the normal vector field n to the embedding is defined by the relation

$$n_\alpha \mathcal{X}^\alpha_{,i} = 0. \quad (5)$$

The indices $\alpha = 0, 1, 2, 3$ and $i = 1, 2, 3$ correspond to the coordinate systems in \mathcal{M} and in Σ respectively. In the Hamiltonian formulation of general relativity the embedding is

required additionally to be space-like with respect to the Lorentzian metric on \mathcal{M} . The defining relation for the normal vector field must then be supplemented by the normalization condition

$$\gamma^{\alpha\beta}n_\alpha n_\beta = -1. \quad (6)$$

The last relation implies that n is a time-like vector field in (\mathcal{M}, γ) when the signature of the Lorentzian metric is $(-1, 1, 1, 1)$.

● **The Lapse Function And The Shift Vector:** For each value of the time parameter t , the deformation vector can be decomposed into two components, one of which lies along the hyper-surface $\mathcal{X}_t(\Sigma)$ and the other of which is parallel to n_t ,

$$\dot{\mathcal{X}}^\alpha = N\gamma^{\alpha\beta}n_\beta + N^i\mathcal{X}^\alpha_{,i}. \quad (7)$$

The functions $N(x, t)$ and $N^i(x, t)$ are known as the lapse function and the shift vector respectively. Their geometric interpretation can be deduced from this relation as follows. The lapse function specifies the proper time separation between the hyper-surfaces $\mathcal{X}_t(\Sigma)$ and $\mathcal{X}_{t+\delta t}(\Sigma)$ measured in the direction normal to the first hyper-surface. The shift vector determines how, for each $x \in \Sigma$, the point $\mathcal{X}_{t+\delta t}(x)$ in \mathcal{M} is displaced with respect to the intersection of the hyper-surface $\mathcal{X}_{t+\delta t}(\Sigma)$ with the normal geodesic drawn from the point $\mathcal{X}_t(x)$.

In order to define the canonical theory, the spacetime metric must be “pulled-back” from \mathcal{M} to $\Sigma \times R$. In local coordinates, the induced spatial metric is written as

$$g_{ij} := \gamma_{\alpha\beta}\mathcal{X}^\alpha_{,i}\mathcal{X}^\beta_{,j}. \quad (8)$$

It is a positive definite tensor of signature (1,1,1) since the embedding is space-like with respect to the Lorentzian structure. The spatial metric contains six of the ten degrees of freedom of the original theory. The normal vector field, the lapse function and the shift vector are the only other quantities that depend on the spacetime metric, so are all candidates for the remaining degrees of freedom. However, if the theory is decomposed with respect to the basis $(n^\alpha, \mathcal{X}^\alpha_i)$, n^α cannot appear in the pulled-back Lagrangian. The reason is that the right sides of equations (5)-(6) consist of pure numbers. The lapse function and the shift vector may therefore be identified with the remaining degrees of freedom and treated as canonical coordinates.

● The Canonical Form Of General Relativity: The canonical theory is obtained by decomposing the Hilbert-Einstein Lagrangian with respect to the spatial metric, the shift vector and the lapse function. A Legendre transformation is performed to replace any time derivative of these variables with their conjugate momenta. The lapse function and the shift vector become non-dynamical Lagrange multipliers. They enforce on the canonical variables the super-Hamiltonian and super-momentum constraints[22],

$$\mathcal{S} = \int d^3x dt [p^{ij} g_{ij} - N \mathcal{H}^{gr}_\perp - N^i \mathcal{H}^{gr}_i], \quad (9)$$

$$\mathcal{H}^{gr}_\perp = \frac{1}{2} g^{-\frac{1}{2}} \left(g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl} \right) p^{ij} p^{kl} - g^{\frac{1}{2}} R \simeq 0, \quad (10)$$

$$\mathcal{H}^{gr}_i = -2 D_j p^j_i \simeq 0. \quad (11)$$

The constraints (10)-(11) satisfy the Dirac algebra[11]

$$\{\mathcal{H}^{gr}_\perp(x), \mathcal{H}^{gr}_\perp(x')\} = g^{ij}(x) \mathcal{H}^{gr}_i(x) \delta_{,j}(x, x') - (x \leftrightarrow x'), \quad (12)$$

$$\{\mathcal{H}^{gr}_\perp(x), \mathcal{H}^{gr}_i(x')\} = \mathcal{H}^{gr}_{\perp,i}(x) \delta(x, x') + \mathcal{H}^{gr}_\perp(x) \delta_{,i}(x, x'), \quad (13)$$

$$\{\mathcal{H}^{gr}_i(x), \mathcal{H}^{gr}_j(x')\} = \mathcal{H}^{gr}_j(x)\delta_{,i}(x, x') - (ix \leftrightarrow jx'). \quad (14)$$

They are first-class since the right hand side of equations (12)-(14) vanishes on the constraint surface (10)-(11). The presence of the spatial metric g^{ij} in equation (12) implies that the Dirac algebra is not a genuine Lie algebra. This becomes a problem in most attempts for a quantum theory of gravity[1,2,12].

Chapter I-3. The Problem Of Time.

- Time In Quantum Theory And In General Relativity: The formulation of quantum theory is grounded on the idea of a measurement made at a particular instant of time. The time parameter is external to the system and belongs to the “classical world” according to the conventional Copenhagen interpretation. The fact that time is not a physical observable is expressed mathematically by the lack of a time operator in the quantum theory[1]. This special property of time applies for non-relativistic quantum theory and for relativistic

particle dynamics as well as for quantum field theory.

However, this view of time cannot be maintained when the theory of general relativity is taken into account. The reason is that the equations of general relativity transform covariantly under changes of spacetime coordinates and physical results remain invariant under such changes. The invariance of the theory under the action of the group of active point transformations, $\text{Diff}(\mathcal{M})$, implies that no intrinsic physical significance can be assigned to the individual mathematical spacetime points. This amounts to the lack of physical observables in the vacuum theory[1]. It also suggests that fundamental features of quantum field theory cannot be maintained, like the notion of a space-like separation, the interpretation of the so-called micro-causality condition and even the canonical commutation relations themselves[1].

● The Approaches To Quantum Gravity: Most approaches to dealing with the contradictory roles assigned to time by the quantum and the classical theory identify time in terms of the internal structure of the system. They can be divided mainly into three categories[1]:

(i) An internal time is identified as a functional of the canonical variables and the constraints are solved, before the system is quantized. The resolved constraints are linear in the momentum conjugate to the internal time, so a linear Schrödinger equation is produced with respect to this choice of time.

(ii) The constraints are imposed at the quantum level as restrictions on the allowed state vectors, and time is identified only after this step. The resulting functional differential equation is known as the Wheeler-DeWitt equation. It is quadratic in the functional derivatives. The notion of time must be recovered from the solutions of this equation, and

the final probabilistic interpretation of the theory must be made after this identification of time.

(iii) The timeless nature of general relativity is preserved. It is assumed that a quantum theory of gravity can be constructed without any mention to the concept of time. The latter is considered to have a status that is purely phenomenological.

Chapter I-4. The Internal-Time Approach.

- The internal-time approach belongs to the first type of quantization schemes. Time is assumed to be hidden among the canonical variables and must be identified and separated before the theory is quantized. The procedure corresponds to the so-called “deparametrization” of general relativity. It is motivated by the parallel that can be drawn between general relativity and parametrized particle dynamics[24].

- Parametrized Particle Dynamics: The canonical action describing the motion of a single

non-relativistic particle of mass m in the potential field $\mathcal{F}(x^i, t)$ has the form[24]

$$S[x^i, p_i] = \int dt \left(p_i \frac{dx^i}{dt} - H(x^i, p_i, t) \right). \quad (15)$$

The precise expression for the Hamiltonian $H(x^i, p_i, t)$ is

$$H(x^i, p_i, t) = \frac{1}{2m} p_i p_i + \mathcal{F}(x^i, t). \quad (16)$$

If the path of the particle is parametrized by an arbitrary label time τ , and the absolute Newtonian time t is adjoined to the configuration variables x^i ,

$$x^\alpha = (t, x^i), \quad x^i = x^i(\tau), \quad p_i = p_i(\tau), \quad t = t(\tau), \quad (17)$$

the action (15) becomes

$$S[x^\alpha(\tau), p_i(\tau)] = \int d\tau \left(p_i(\tau) \frac{dx^i(\tau)}{d\tau} - H[x^\alpha(\tau), p_i(\tau)] \frac{dt(\tau)}{d\tau} \right). \quad (18)$$

Expression (18) is numerically equal to the original expression (15) so its variation with respect to the canonical variables x^i and p_i gives the correct equations of motion. In addition, the variation with respect to the Newtonian time t yields

$$\frac{dH(\tau)}{d\tau} = \frac{\partial H(t)}{\partial t} \frac{dt(\tau)}{d\tau}, \quad (19)$$

which is valid by virtue of the original Hamilton equations for x^i and p_i . The parametrized action (18) is therefore consistent.

The Hamiltonian process requires the definition of the momentum p_0 conjugate to the Newtonian time t ,

$$p_0 := -H[x^\alpha, p_i]. \quad (20)$$

The canonical action then becomes

$$S[x^\alpha(\tau), p_\alpha(\tau)] = \int d\tau \left(p_\alpha(\tau) \frac{dx^\alpha(\tau)}{d\tau} - \mathcal{N} \mathcal{H}[x^\alpha(\tau), p_\alpha(\tau)] \right). \quad (21)$$

Notice that a new quantity p_α has been introduced, defined by

$$p_\alpha = (p_0, p_i), \quad p_0 = p_0(\tau). \quad (22)$$

The geometric meaning of \mathcal{N} is recovered by varying this action with respect to p_0 ,

$$\mathcal{N} = \frac{dt(\tau)}{d\tau}. \quad (23)$$

The variation with respect to the non-dynamical Lagrange multiplier \mathcal{N} enforces on the canonical data the constraint

$$\mathcal{H}[x^\alpha(\tau), p_\alpha(\tau)] = p_0(\tau) + H[x^\alpha(\tau), p_i(\tau)] = 0. \quad (24)$$

Finally, the variation with respect to the remaining variables leads to valid equations by virtue of the original theory.

- **The Internal Time Formalism:** The parallel that can be drawn between this procedure and canonical geometrodynamics is the following. The parametrized Hamiltonian in (21) is constrained to vanish, the Newtonian time behaves as an one-dimensional embedding, and the Lagrange multiplier enforcing the constraint is the analogue of the lapse function. The observation is that the constraint $\mathcal{H} = 0$ arises in the parametrized model because the Newtonian time t and its conjugate momentum p_0 have been adjoined to the true dynamical variables x^i, p_i . It could be the case that the action for geometrodynamics is already in parametrized form and can even be de-parametrized; that is, reduced to its basic true degrees of freedom.

The conjecture is that there exists a canonical transformation[25],

$$\left(g_{ij}(x), p^{ij}(x)\right) \quad to \quad \left(X^A(x), P_A(x), \phi^q(x), \pi_q(x)\right), \quad (25)$$

that separates the four embedding variables $X^A(x)$ that specify the hyper-surface from the two true gravitational degrees of freedom $\phi^q(x)$ ($A = 0, 1, 2, 3$, $q = 1, 2$). The constraint equations (10)-(11) are then replaced by the equivalent set

$$H_A(x) = P_A(x) + h_A(x; X, \phi, \pi] = 0. \quad (26)$$

The modified constraints (26) correspond to equation (24) valid for the parametrized particle. Drawing the analogy even further, the quantity $h_A(x; X, \phi, \pi]$ is interpreted as the energy density and the energy flux carried by the variables $\phi^q(x)$ and $\pi_q(x)$ through the hyper-surface $X^A(x)$. In the form (26), the constraints are imposed on the physical states according to the Dirac quantization algorithm. Because of their linearity they lead to a first-order Schrödinger equation. If this separation between embedding variables and true degrees of freedom can be achieved, a considerable progress towards a quantum theory of gravity will have been made.

● **Problems With The Internal Time Formalism:** However, besides the standard technical issues associated with a Schrödinger-type equation[1], the internal time approach faces problems that render it almost unattainable:

1. **Global problem.** Calculations performed in simple models have shown that the new system of constraints (26) cannot be made globally equivalent to the original system (10)-(11). It is reasonable to surmise that this problem becomes worse in the case of full geometrodynamics.

2. Multiple choice problem. If a natural choice of internal time does not exist, the ensuing quantum theories must either be equivalent or, at least, related in a specific way. Simple examples show that this is not the case.
3. Spacetime problem. What is implied in the internal-time program is that the time coordinate should be constructed purely of the canonical data. Such a time must be independent of the particular foliation relative to which the canonical formalism has been defined, so it can only be a spacetime scalar. Assuming only locality, it has been shown that scalar internal-time functionals do not exist[2].

Chapter I-5. The Gaussian Time Formulation.

- The probable failure of the internal-time approach suggests that the analogy drawn between geometrodynamics and parametrized dynamics is a misleading one. Nevertheless, this analogy motivated an alternative approach to quantum gravity, where the framework

for dealing with the conceptual aspects of time arises naturally.

- **Coordinate Conditions:** Isham and Kuchař[26] discussed the issue of representing spacetime diffeomorphisms in canonical general relativity. They needed to construct a homomorphic mapping of spacetime vector fields into the Poisson bracket algebra of the geometrodynamical phase space. Contrary to the internal time scheme, they did not consider canonical gravity as being already parametrized but, instead, they parametrized it once more. The geometrodynamical phase space is extended by the space of embeddings of the spatial manifold Σ in the spacetime \mathcal{M} and, therefore, the required homomorphic mapping is constructed.

When extending the phase space, the need for consistency led them to restrict the spacetime metrics by Gaussian coordinate conditions with respect to an auxiliary foliation structure. The constraints (10)-(11) are suspended temporarily and are re-introduced later, after the embedding canonical variables have been adjoined to the gravitational ones. This is achieved through varying the auxiliary structure.

The procedure of breaking the invariance of general relativity, and restoring it again by parametrization, leads to the modification of the constraints by terms which are linear in the momenta conjugate to the Gaussian coordinates. As in the case of an internal time approach, the equation obtained by imposing the new constraints as a restriction on the physical states is a Schrödinger equation. The ensuing theory can be viewed as vacuum quantum gravity but it can also be criticized as lacking physical interpretation[2].

- **The Reference fluid:** Addressing this issue, Kuchař and Torre[27] gave a phenomenological interpretation for the Gaussian conditions by taking them into account through

a different technical procedure. The Gaussian coordinates are adjoined to the Hilbert-Einstein action by Lagrange multipliers and the total action is varied. The additional variables introduce a source term into Einstein's field equations and are interpreted as a reference fluid. The canonical analysis of the fluid results in a set of constraints similar to that in [26], and a Schrödinger equation is obtained through the Dirac quantization scheme.

The main advantages of the Gaussian-time approach over the internal-time one are the following:

- (i) The fluid variables are spacetime scalars by construction, so there is no spacetime problem.
- (ii) The introduction of the reference system is associated with a privileged time so there is no multiple choice problem, either.

Unfortunately, although these gains are considerable, the Gaussian reference fluid suffers from a basic problem. Its energy-momentum tensor does not satisfy the energy conditions of general relativity, so the fluid can only be given an interpretation that is phenomenological.

Part II:

A New Lie Algebra For Vacuum General Relativity

Chapter II-1. The Discovery Of A Genuine Lie Algebra.

● Incoherent dust: In their search for a realistic medium for the reference fluid, Brown and Kuchař[13] avoided any mention to coordinate conditions. Instead, they constructed a physical Lagrangian that describes a globally hyperbolic spacetime filled with incoherent dust,

$$\mathcal{S} = \int d^4X \left(-\frac{1}{2}\right) |\gamma|^{\frac{1}{2}} M \left(\gamma^{\alpha\beta} U_\alpha U_\beta + 1 \right). \quad (27)$$

The four-velocity U_α of the dust is defined by its decomposition in the co-basis $Z^K_{,\alpha}$,

$$U_\alpha = -T_{,\alpha} + W_i Z^i_{,\alpha}. \quad (28)$$

The scalars $Z^K = (T, Z^i)$ are assumed to be four independent functions of the spacetime coordinates. The values of the variables Z^i correspond to the co-moving coordinates of the dust particles, and the value of the variable T corresponds to the proper time measured along the particle flow lines. The three spatial components W_i of the four-velocity in the dust frame $\{Z^i\}$ and the multiplier M are all state variables, whose physical interpretation follows from the ensuing equations of motion[13].

The co-moving coordinates of the dust particles and the proper time along the dust world-lines are treated as canonical coordinates, so a privileged dynamical reference frame and time foliation are introduced into spacetime. Disregarding certain problems concerning the factor ordering[1,2], the Dirac quantization of the coupled system provides an improved phenomenological approach to the problem of time in quantum gravity. The work of Brown and Kuchař is the starting point of this thesis.

• Self-Commuting Combinations: While studying the canonical decomposition of the dust action, the authors of [13] came across a weight-two scalar combination of the gravitational constraints,

$$G(x) := \mathcal{H}^{gr^2}_{\perp}(x) - g^{ij}\mathcal{H}^{gr}_i(x)\mathcal{H}^{gr}_j(x). \quad (29)$$

The Poisson brackets of $G(x)$ with itself vanish strongly. If $G(x)$ replaces the usual Hamiltonian constraint to form an equivalent set of constraints for *vacuum* general relativity,

$$G(x) = 0 = \mathcal{H}^{gr}_i(x), \quad (30)$$

a genuine Lie algebra is created.

The new algebra takes the form

$$\{G(x), G(x')\} = 0, \quad (31)$$

$$\{G(x), \mathcal{H}^{gr}_i(x')\} = G_{,i}(x)\delta(x, x') + 2G(x)\delta_{,i}(x, x'), \quad (32)$$

$$\{\mathcal{H}^{gr}_i(x), \mathcal{H}^{gr}_j(x')\} = \mathcal{H}^{gr}_j(x)\delta_{,i}(x, x') - (ix \leftrightarrow jx'). \quad (33)$$

It corresponds to the semi-direct product of the Abelian algebra generated by $G(x)$, equation (31), with the algebra of spatial diffeomorphisms $\text{LDiff}\Sigma$ generated by $\mathcal{H}^{gr}_i(x)$, equation (33). The Poisson bracket (32) reflects the transformation of $G(x)$, under $\text{Diff}\Sigma$, as a scalar density of weight two.

A similar result was obtained by Kuchař and Romano[14]. They coupled gravity to a massless scalar field and extracted another weight-two scalar combination of the gravitational constraints,

$$\Lambda_{\pm}(x) := g^{\frac{1}{2}}(x) \left(-\mathcal{H}^{gr}_{\perp}(x) \pm \sqrt{G(x)} \right). \quad (34)$$

These results are significant because in the presence of a genuine Lie algebra some of the problems associated with quantization can be eliminated. This applies particularly to a group-oriented approach, where the Hilbert space of the quantum theory is constructed by studying the representations of a group of observables, that often include symmetries of the classical system[1]. The standard super-Hamiltonian and super-momentum constraints are such observables, but they do not form a Lie group since their closing relations under the Poisson bracket operations do not produce a genuine Lie algebra. The presence of a genuine algebra allows the definition of a group and, hence, the use of powerful techniques from group representation theory for the construction of the appropriate Hilbert space.

● The Weight- ω Equation: The issue that arises is whether these self-commuting combinations convey any general message about the structure of canonical general relativity[14]. An advance towards understanding their nature was made by Markopoulou[16]. She constructed a nonlinear partial differential equation satisfied by scalar combinations of the gravitational constraints that close according to the Abelian algebra (31)-(33).

The main observation was that any scalar density \mathcal{W}_ω of arbitrary-weight can be written in terms of simpler combinations of the constraints and of the spatial metric,

$$\mathcal{W}_\omega(x) = g^{\frac{\omega}{2}}(x)W_\omega[h(x), f(x)], \quad (35)$$

assuming that \mathcal{W}_ω is an ultra-local function of them. The parameter ω denotes the weight of the corresponding scalar densities. The basic combinations of the constraints are defined by

$$h(x) := g^{-\frac{1}{2}}(x)\mathcal{H}^{gr}_\perp(x), \quad (36)$$

$$f(x) := g^{-1}(x)g^{ij}(x)\mathcal{H}^{gr}_i(x)\mathcal{H}^{gr}_j(x) \quad (37)$$

and transform as scalar densities of weight zero. Notice that both G and Λ_{\pm} can be written in the form (35) for weight two.

The requirement that the Poisson brackets of $\mathcal{W}_{\omega}(x)$ should vanish strongly results in a differential equation for $W_{\omega}(x)$,

$$\frac{\omega}{2}W_{\omega}(x)W_{\omega f}(x) = f(x)W_{\omega f}^2(x) - \frac{1}{4}W_{\omega h}^2(x), \quad (38)$$

where the notation $W_{\omega h} := \frac{\partial W_{\omega}}{\partial h}$ and $W_{\omega f} := \frac{\partial W_{\omega}}{\partial f}$ has been used.

Its general solution can be found exactly, and is given[16] by

$$\begin{aligned} W_{\omega}[h, f, B(\alpha[h, f])] &= \pm \left[\left(h - \frac{1}{2}B'(\alpha[h, f]) \right) + \sqrt{\left(h - \frac{1}{2}B'(\alpha[h, f]) \right)^2 - f} \right]^{\frac{\omega}{2}} \\ &\times \exp \left(B(\alpha[h, f]) + \frac{\omega}{2} \frac{\frac{1}{2}B'(\alpha[h, f])}{\sqrt{\left(h - \frac{1}{2}B'(\alpha[h, f]) \right)^2 - f}} \right). \end{aligned} \quad (39)$$

The form of the function $\alpha[h, f]$ is determined by solving algebraically the equation

$$\alpha = - \frac{\omega}{4\sqrt{\left(h - \frac{1}{2}B'(\alpha) \right)^2 - f}} \quad (40)$$

for a given choice of $B(\alpha)$. Complex solutions for $W_{\omega}(x)$ can exist.

● A Possibility: Expressions (39) and (40) are based on algebraic considerations, so their physical relevance is not clear. An insight into their origin could be gained if they were shown to be related to systems similar to the ones discussed in [14] and [14]. In particular, the actions for dust and for a massless scalar field could arise as different versions of a general action, parametrized by an arbitrary function of one variable. This possibility is supported by the fact that the general solution (39)-(40) has a similar dependence upon

such a function. It is also compatible with the general properties of first-order partial differential equations[28].

An obstacle to this construction is that the fields used in [13] and in [14] are unequal in number. However, it can be shown that the relevant results G and Λ_{\pm} depend only on the form of the action and not on the number of the canonical fields. An action of a single field could therefore suffice, provided that it is parametrized by an arbitrary function of one variable and reduces to the form of the actions in [13] and [14] for particular choices of this function. When coupled to gravity, it could provide the general solution of equation (38) for weight two.

In addition, if the weight-two procedure proved to be successful it would be extended trivially to an arbitrary weight. This follows from a remarkable property of equation (38).

If W_{ω} is a solution of weight ω then $W_{\omega'}$, defined by

$$W_{\omega'} := W_{\omega}^{\frac{\omega'}{\omega}}, \tag{41}$$

is a new solution of weight ω' . Notice that both ω and ω' must be different from zero so that the algorithm is well-defined and invertible.

Chapter II-2. Treating The Algebra Algebraically.

Equation (38) is related to a generalized action that has the properties described above. As it stands, the equation does not make this connection clear, so an ansatz is used to convert it into an appropriate form. The ansatz is parametrized by the weight ω appearing in equation (38) and expresses W_ω in terms of two functions λ and μ . Like W_ω they are ultra-local functions of the basic combinations h and f and they transform as scalar densities of weight zero. The ansatz transforms the nonlinear equation (38) into a pair of coupled quasi-linear partial differential equations for λ and μ , called the “linear” equation.

● Preliminary Remarks About The ω -Equation: If any of the partial derivatives of W_ω is trivial, equation (38) implies that W_ω is either a function of f , alone, or a constant. In both cases, the information concerning the Hamiltonian constraint is lost. These special solutions have been excluded from the following discussion, although the reasons for excluding them will arise later, in chapters II-4 and II-5.

Recall, also, that equation (38) allows the existence of complex solutions which, usually, cannot be reconciled to the idea of a physical system. This is particularly true here, since these solutions are required later to be positive definite. However, a complex combination of the gravitational constraints is not necessarily complex-valued. For example, the weight-one solution $i\sqrt{h^2 - f}$ is positive in a region of the phase space where $f > h^2$. This particular region is not accessible to vacuum general relativity, but may be so to a coupled system. It is therefore preferable to accept all solutions of equation (38) at this stage and

make the necessary amendments later, in chapter II-4.

● The ω -Ansatz And The Linear Equation: The one-parameter family of “ansatzes” has the following form,

$$W_\omega[h, f] = \lambda^{\frac{\omega}{2}}[h, f] \left(h - \mu[h, f] + \sqrt{(h - \mu[h, f])^2 - f} \right)^{\frac{\omega}{2}}. \quad (42)$$

Each ω -ansatz transforms the corresponding ω -equation (38). Both signs for the square root are permitted. This is not denoted by a \pm sign in order to keep the notation simple. The square root in (42) is denoted by the letter R , and the square-bracket notation is dropped,

$$R := \sqrt{(h - \mu)^2 - f}. \quad (43)$$

The partial derivatives of W_ω are expressed in terms of λ , μ and R according to

$$\begin{aligned} W_{\omega H} &= \frac{\omega}{2} \lambda^{\frac{\omega}{2}} (h - \mu + R)^{\frac{\omega}{2}} \left(\frac{1}{\lambda} \lambda_h - \frac{1}{R} \mu_h + \frac{1}{R} \right), \\ W_{\omega F} &= \frac{\omega}{2} \lambda^{\frac{\omega}{2}} (h - \mu + R)^{\frac{\omega}{2}} \left(\frac{1}{\lambda} \lambda_f - \frac{1}{R} \mu_f - \frac{1}{2R(h - \mu + R)} \right). \end{aligned} \quad (44)$$

When expressions (42)-(44) are used in equation (38), this becomes:

$$\left(\frac{1}{\lambda} \lambda_f - \frac{1}{R} \mu_f \right) \left[-f \left(\frac{1}{\lambda} \lambda_f - \frac{1}{R} \mu_f \right) + \frac{h - \mu}{R} \right] + \frac{1}{4} \left(\frac{1}{\lambda} \lambda_h - \frac{1}{R} \mu_h \right) \left[\left(\frac{1}{\lambda} \lambda_h - \frac{1}{R} \mu_h \right) + \frac{2}{R} \right] = 0. \quad (45)$$

Noticeably, the arbitrary weight ω has been eliminated.

There exist four obvious solutions of equation (45), corresponding to four different pairs of coupled quasi-linear equations for μ and λ :

$$\frac{1}{\lambda} \lambda_f - \frac{1}{R} \mu_f = 0 \quad \text{and} \quad \frac{1}{\lambda} \lambda_h - \frac{1}{R} \mu_h = 0, \quad (46)$$

$$\frac{1}{\lambda} \lambda_f - \frac{1}{R} \mu_f = 0 \quad \text{and} \quad \frac{1}{\lambda} \lambda_h - \frac{1}{R} \mu_h = -\frac{2}{R}, \quad (47)$$

$$\frac{1}{\lambda}\lambda_f - \frac{1}{R}\mu_f = \frac{h - \mu}{R} \quad \text{and} \quad \frac{1}{\lambda}\lambda_h - \frac{1}{R}\mu_h = 0, \quad (48)$$

$$\frac{1}{\lambda}\lambda_f - \frac{1}{R}\mu_f = \frac{h - \mu}{R} \quad \text{and} \quad \frac{1}{\lambda}\lambda_h - \frac{1}{R}\mu_h = -\frac{2}{R}. \quad (49)$$

Given a μ , any of the above pairs of equations can be solved for the corresponding λ , provided that the system of the two partial equations for λ is not contradictory. Then, the ω -ansatz (42) can be used to produce solutions W_ω of the corresponding weight. An equivalent procedure can be followed if a λ is given initially.

Furthermore, the above pairs are equivalent, in the sense that for each weight they all lead to the same family of solutions of the non-linear equation (38). The proof can be found in Appendix A. The most symmetric of the equivalent pairs, equation (46), is then singled out. It is called the “linear” equation and is the one related directly to the action principle. Its solutions must be compared with the general solution of each ω -equation (38). The surprising result is that, for all weights, equations (38) and (46) are equivalent.

In particular, given a solution W_ω of the weight- ω equation, there exist unique functions

$$\bar{\lambda} = \frac{-2W_\omega^{\frac{2}{\omega}}W_{\omega f}}{W_{\omega h}}, \quad (50)$$

$$\bar{\mu} = h + \frac{W_{\omega f}}{W_{\omega h}}f + \frac{1}{4}\frac{W_{\omega h}}{W_{\omega f}}, \quad (51)$$

$$\bar{R} = \frac{W_{\omega f}}{W_{\omega h}}f - \frac{1}{4}\frac{W_{\omega h}}{W_{\omega f}}, \quad (52)$$

that satisfy the linear equation and lead to W_ω through the corresponding ω -ansatz. The derivation of equations (50)-(52) can be found in Appendix B. The over-bar symbol is a reminder of the uniqueness of these expressions. To be precise, the expression for $\bar{\lambda}$ is not exactly unique but holds up to an $\frac{\omega}{2}$ power of unity. Notice that equations (50)-(52)

are well-defined in general, because $W_{\omega h}$ and $W_{\omega f}$ have been required to be non-trivial functions of h and f . Of course, the phase space should be restricted to those regions where also the numerical values of $W_{\omega h}$ and $W_{\omega f}$ are non-trivial.

Chapter II-3. Actions Leading To The Algebra.

- The relevant action involves a scalar field with a non-derivative coupling to gravity and, initially, two arbitrary functions of a Lagrange multiplier. It is the simplest action that possesses the parametrization by an arbitrary function of one variable and includes the actions in [13] and [14] as sub-cases. The required parametrization arises only after the elimination of the non-dynamical multiplier. Because there is no detailed reference to an underlying physical interpretation the following construction should be viewed mainly as a mathematical one.

- The Scalar Field Action: The action functional S^ϕ is introduced as

$$S^\phi[\phi, M, \gamma^{\alpha\beta}] = \int d^4X |\gamma|^{\frac{1}{2}} \left(\frac{1}{2} \lambda(M) \gamma^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \mu(M) \right). \quad (53)$$

The dependence of the fields on the spacetime point X is not written explicitly. The functions $\lambda(M)$ and $\mu(M)$ are continuous functions of the Lagrange multiplier M . They are fixed (i.e., non-canonical) but otherwise arbitrary. Since the scalar field has to be present in the action functional, $\lambda(M)$ is required to be different from zero. No such restriction is imposed on $\mu(M)$. Notice that the notation for the two functions of the multiplier reflects the notation used in chapter II-2.

Keeping the conventional dimension of inverse length for the scalar field, a consistent attribution of dimensions to the various terms appearing in (53) is the following:

$$[\phi] = L^{-1}, \quad [M] = [\lambda(M)] = L^0 = 1, \quad [\mu(M)] = L^{-4}. \quad (54)$$

This means that $\mu(M(X))$ should be considered as a function of the multiplier $M(X)$ scaled by a constant scalar function $C(X)$,

$$\mu(M(X)) = C(X)\rho(M(X)). \quad (55)$$

The dimensions of the new functions are $[C] = L^{-4}$ and $[\rho(M)] = L^0 = 1$. For simplicity, appropriate units can be assumed so that $C(X) = 1$, in which case ρ may be identified with μ .

● The Hamiltonian Analysis Of The Coupled System: The scalar field action (53) is coupled to the gravitational Einstein-Hilbert action $S^{gr}[\gamma_{\alpha\beta}]$,

$$S^{gr}[\gamma_{\alpha\beta}] = \int d^4X |\gamma|^{\frac{1}{2}} R[\gamma_{\alpha\beta}]. \quad (56)$$

The canonical analysis of the total action,

$$S^T := S^{gr} + S^\phi, \quad (57)$$

results in the coupled constraints[9]

$$\mathcal{H}_\perp^T := \mathcal{H}^{gr}_\perp + \mathcal{H}_\perp^\phi = 0, \quad (58)$$

$$\mathcal{H}_i^T := \mathcal{H}^{gr}_i + \mathcal{H}_i^\phi = 0. \quad (59)$$

Their form is common to any theory with a non-derivative coupling to gravity[5].

The gravitational parts of the constraints, \mathcal{H}^{gr}_\perp and \mathcal{H}^{gr}_i , are identical to the constraints of vacuum general relativity written out in equations (10) and (11). The scalar field contributions \mathcal{H}_\perp^ϕ and \mathcal{H}_i^ϕ are given by

$$\mathcal{H}_i^\phi = \pi\phi_{,i}, \quad (60)$$

$$\mathcal{H}_\perp^\phi = g^{\frac{1}{2}} \left(-\frac{1}{2} \frac{\pi^2}{g\lambda(M)} - \mu(M) - \frac{1}{2} \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}_i^\phi \mathcal{H}_j^\phi \right). \quad (61)$$

Notice that the kinetic energy of the scalar field has to be positive. Equation (61) then implies that $\lambda(M)$ must be negative. On the other hand, $\mu(M)$ appears as a cosmological constant in equation (61) so it may take any real value.

● The Two Equations For M And π : At this stage, the total action S^T is varied with respect to the multiplier. The latter appears only in the super-Hamiltonian for the scalar field, so the variation results in the following condition

$$\frac{d\mathcal{H}_\perp^\phi}{dM} = 0. \quad (62)$$

Equation (62) can be written equivalently as

$$\frac{1}{2} \frac{\pi^2}{g\lambda^2(M)} \lambda'(M) - \mu'(M) - \frac{1}{2} \frac{1}{\pi^2} \lambda'(M) g^{ij} \mathcal{H}_i^\phi \mathcal{H}_j^\phi = 0, \quad (63)$$

where $\lambda'(M)$ and $\mu'(M)$ denote the total derivatives of $\lambda(M)$ and $\mu(M)$ with respect to M .

The constraints (58) and (59) can now be used to re-express equations (61), (63) in terms of the gravitational contributions to these constraints,

$$\frac{1}{2} \frac{\pi^2}{g\lambda(M)} + \frac{1}{2} \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}^{gr}_i \mathcal{H}^{gr}_j = h - \mu(M), \quad (64)$$

$$\frac{1}{2} \frac{\pi^2}{g\lambda^2(M)} \lambda'(M) - \frac{1}{2} \frac{1}{\pi^2} g^{ij} \mathcal{H}^{gr}_i \mathcal{H}^{gr}_j \lambda'(M) = \mu'(M). \quad (65)$$

The quantity h is the scalar density of weight zero defined in equation (36). The aim is to solve equations (64)-(65) for π and M in terms of \mathcal{H}^{gr}_\perp and \mathcal{H}^{gr}_i . Because the solution depends on the actual form of the derivatives, some special cases must be considered separately.

● Solving The Two Equations For M And π . The General Case: This occurs when both the derivatives of λ and μ are non-trivial,

$$\lambda'(M) \neq 0 \quad \mu'(M) \neq 0. \quad (66)$$

If this condition holds, equation (65) can be multiplied by $\lambda(M)/\lambda'(M)$, resulting in the equivalent relation

$$\frac{1}{2} \frac{\pi^2}{g\lambda(M)} - \frac{1}{2} \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}^{gr}_i \mathcal{H}^{gr}_j = \frac{\mu'(M)\lambda(M)}{\lambda'(M)}. \quad (67)$$

Equations (64) and (67) must be solved for π and M in terms of the gravitational contributions to the constraints. This can be done by adding and subtracting (64) and (67), and then cross-multiplying the resulting equations to eliminate the field momenta. An algebraic equation arises that determines the multiplier M as a function of \mathcal{H}^{gr}_\perp and \mathcal{H}^{gr}_i , alone,

$$\frac{\mu'(M)\lambda(M)}{\lambda'(M)} = \sqrt{\left(h - \mu(M)\right)^2 - f}. \quad (68)$$

The quantity f is the one defined in equation (37). Both signs for the square root are allowed in (68), although this is not denoted explicitly.

The corresponding expression for the field momentum as a function of \mathcal{H}^{gr}_\perp and \mathcal{H}^{gr}_i , alone, is given by

$$\begin{aligned} \frac{1}{(g^{\frac{1}{2}})^2} \pi^2 [\mathcal{H}^{gr}_\perp \mathcal{H}^{gr}_i] &= \lambda(M[\mathcal{H}^{gr}_\perp, \mathcal{H}^{gr}_i]) \times \\ &\times \left(h - \mu(M[\mathcal{H}^{gr}_\perp, \mathcal{H}^{gr}_i]) + \sqrt{\left(h - \mu(M[\mathcal{H}^{gr}_\perp, \mathcal{H}^{gr}_i]) \right)^2 - f} \right). \end{aligned} \quad (69)$$

Equations (68) and (69) are the required solutions of the original system of equations (64) and (65).

The observation is that the solutions $M[\mathcal{H}^{gr}_\perp, \mathcal{H}^{gr}_i]$ and $\pi^2[\mathcal{H}^{gr}_\perp, \mathcal{H}^{gr}_i]$ can be written solely in terms of the scalar combinations h and f . This follows directly from the actual form of equations (68) and (69). In addition, λ and μ can be regarded as sole functions of h and f as well, according to

$$\lambda[h, f] := \lambda(M[\mathcal{H}^{gr}_\perp, \mathcal{H}^{gr}_i]), \quad \text{and} \quad \mu[h, f] := \mu(M[\mathcal{H}^{gr}_\perp, \mathcal{H}^{gr}_i]). \quad (70)$$

As a result, equation (69) can be put into the equivalent form

$$\frac{1}{(g^{\frac{1}{2}})^2} \pi^2[h, f] = \lambda[h, f] \left(h - \mu[h, f] + R[h, f] \right). \quad (71)$$

Notice that the definition (43) for the function R has been used in (71).

If the above expression is raised to the power of $\frac{\omega}{2}$ it can be recognized as the ω -ansatz written out in equation (42). It satisfies the differential equation (38) provided that $\lambda[h, f]$, $\mu[h, f]$ and the square root in equation (71) satisfy the common to all weights linear equation (46). The following argument shows that this is true indeed.

Since (68) is an algebraic equation for M , it must hold identically when written in terms of an actual solution $M[h, f]$. Therefore, it becomes a differential equation for $\lambda(M[h, f]) \equiv \lambda[h, f]$ and $\mu(M[h, f]) \equiv \mu[h, f]$, regardless of the particular form of the functions $\lambda(M)$ and $\mu(M)$. Furthermore, equation (68) makes certain that its solution $M[h, f]$ satisfies the conditions

$$M_h \neq 0 \quad \text{and} \quad M_f \neq 0. \quad (72)$$

Equation (68) can be then multiplied by M_h and M_f , resulting in the following pair of partial differential equations for $\mu[h, f]$ and $\lambda[h, f]$,

$$\frac{1}{\lambda} \lambda_h - \frac{1}{\sqrt{(h - \mu)^2 - f}} \mu_h = 0 \quad \text{and} \quad \frac{1}{\lambda} \lambda_f - \frac{1}{\sqrt{(h - \mu)^2 - f}} \mu_f = 0. \quad (73)$$

This is precisely the linear equation (46). Notice that the gravitational phase space must be restricted to the regions where the quantity inside the square root as well as the whole right side of equation (71) are positive.

● Solving The Two Equations For M And π . The Special Cases: Returning to equations (64) and (65), the special cases are considered. There are three possibilities:

(i) The Kuchař-Romano family. This occurs when both the derivatives of $\lambda(M)$ and $\mu(M)$ are trivial,

$$\lambda'(M) = 0 \quad \text{and} \quad \mu'(M) = 0. \quad (74)$$

This means that $\lambda(M)$ and $\mu(M)$ are constant functions

$$\mu(M) = C_1 \quad \lambda(M) = C_2. \quad (75)$$

They are required to be real and negative respectively.

In this case, there is no multiplier present in the total action. Equation (65) is then satisfied trivially, both sides being equal to zero. Accordingly, the coupled system of equations (64), (65) for M and π^2 reduces to the single equation (64). The latter expresses π^2 directly as a function of h and f , according to

$$\frac{\pi^2}{(g^{1/2})^2} = C_2 \left((h - C_1) + \sqrt{(h - C_1)^2 - f} \right). \quad (76)$$

When raised to an $\frac{\omega}{2}$ power, equation (76) is recognized as the ω -ansatz. The required identification is

$$\lambda[h, f] \equiv C_2, \quad (77)$$

$$\mu[h, f] \equiv C_1. \quad (78)$$

The linear equation is satisfied trivially for these $\lambda[h, f]$ and $\mu[h, f]$, and therefore expression (76) provides further solutions of the differential equation (38). They are all required to be positive. Notice that this case reduces to the Kuchař-Romano combination under the identification $\omega = 2$, $C_1 = 0$ and $C_2 = -1$.

(ii) The Pseudo-multiplier. This case occurs when

$$\lambda'(M) = 0 \quad \text{and} \quad \mu'(M) \neq 0. \quad (79)$$

This implies that λ is a constant function C_2 , which is required to be negative. Equation (65) then becomes

$$\mu'(M) = 0. \quad (80)$$

Notice that equations (79) and (80) are not contradictory. The first implies that $\mu(M)$ is

a non-trivial function of M , while the second is an algebraic equation for determining M provided that the non-trivial function $\mu(M)$ is given.

Any solution of equation (80) can yield only a numerical value for M . Equivalently, M is not a proper multiplier but merely “fixes itself a value”. However, equation (64) can still be solved for π^2 , leading to

$$\frac{\pi^2}{(g^{1/2})^2} = -C_2 \left((h - C_1) + \sqrt{(h - C_1)^2 - f} \right). \quad (81)$$

The constant function C_1 corresponds to the real numerical value of $\mu(M)$ after the elimination of the pseudo-multiplier. Therefore, case (ii) is essentially identical to case (i).

(iii) The Null-Vector Family. This occurs when

$$\lambda'(M) \neq 0 \quad \text{and} \quad \mu'(M) = 0. \quad (82)$$

The function $\mu(M)$ is equal to a constant function C_1 , which is required to be real. Equation (65) then becomes

$$\left(\frac{\pi^2}{g\lambda(M)} - \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}^{gr}_i \mathcal{H}^{gr}_j \right) \frac{\lambda'(M)}{\lambda(M)} = 0. \quad (83)$$

As a result, M either takes a real numerical value—thus producing exactly the same combinations as in special cases (i) and (ii)—or satisfies the condition

$$\frac{\pi^2}{g\lambda(M)} - \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}^{gr}_i \mathcal{H}^{gr}_j = 0. \quad (84)$$

When equation (84) is combined with equation (64), it leads to a constraint between the two variables f and h ,

$$f = (h - C_1)^2. \quad (85)$$

This is the only case where π^2 drops out of its defining equations (64)-(65). However, equation (85) still leads to a self-commuting combinations of weight zero when solved in terms of the constant C_1 . An example is when $\lambda(M) = M$ and $\mu(M) = 0$. It can be interpreted as a coordinate condition on $\gamma_{\alpha\beta}$ such that the non-dynamical field ϕ becomes null,

$$\gamma^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} = 0. \tag{86}$$

A corresponding physical system can be interpreted as null-dust.

Chapter II-4. The Inverse Procedure.

● **Actions For Given Solutions:** It has been shown that for all choices of $\lambda(M)$ and $\mu(M)$ the action functional (53) yields solutions of the ω -equation. The converse statement is also true. Every solution of equation (38) can be derived from an action principle of the form (53). For the present purposes it suffices that only a limited part of this statement is

proved.

In particular, it has to be shown that there exist some functions $M[h, f]$, $\mu(M)$ and $\lambda(M)$ that satisfy the conditions

$$\mu(M[h, f]) = \bar{\mu}[h, f], \quad (87)$$

$$\lambda(M[h, f]) = \bar{\lambda}[h, f], \quad (88)$$

$$\sqrt{(h - \mu(M[h, f]))^2 - f} = \bar{R}[h, f], \quad (89)$$

provided that $\bar{\mu}$, $\bar{\lambda}$ and the corresponding W_ω are real, negative and positive-valued respectively. The over-bar symbol indicates the uniqueness of these expressions for each given choice of $W_\omega[h, f]$, as explained in chapter II-2.

Notice that if a solution of equations (87)-(89) exists then the problems mentioned in chapter II-2 concerning trivial and complex combinations no longer apply, since they are eliminated by the assumption of existence of such a solution. This justifies the choices made at the beginning of chapter II-2 concerning the subsequent treatment of these combinations. Returning to equations (87)-(89), it may be observed that the linear equation (46) makes certain that $\bar{\lambda}[h, f]$ and $\bar{\mu}[h, f]$ can either be functions of both h and f or constants. If this is not the case, the system of the two partial differential equations in (46) is self-contradictory. Therefore, there are two cases that must be considered separately:

(i) Constant functions: If $\bar{\mu}$ and $\bar{\lambda}$ are negative and real constants, C_1 and C_2 respectively, then the required solution can be found without needing to specify the form of the function $M[h, f]$. This follows from the results concerning the special cases (i)-(iii) in chapter II-3.

Specifically, the required functions $\lambda(M)$ and $\mu(M)$ can be identified as

$$\lambda(M) = \bar{\lambda}[h, f] = C_2, \quad \text{and} \quad \mu(M) = \bar{\mu}[h, f] = C_1. \quad (90)$$

The above relations satisfy equation (89) for an appropriate choice of sign of the square root and, therefore, the problem of finding an action functional is solved.

(ii) Non-trivial functions: When $\bar{\mu}[h, f]$ and $\bar{\lambda}[h, f]$ are, respectively, real and negative non-trivial functions of h and f the situation is more complicated. The observation is that the Jacobian of $\bar{\mu}[h, f]$ and $\bar{\lambda}[h, f]$ with respect to h and f is identically zero. This follows directly from the form of the linear equation (46), and implies that $\bar{\mu}[h, f]$ and $\bar{\lambda}[h, f]$ are functionally dependent. As a result, there exist at least local regions of the gravitational phase space where $\bar{\mu}[h, f]$ can be solved as a unique and real function of $\bar{\lambda}[h, f]$,

$$\bar{\mu}[h, f] = \kappa(\bar{\lambda}[h, f]). \quad (91)$$

The reason that the above is true is because $\bar{\lambda}$ and $\bar{\mu}$ are real-valued. Since $\bar{\mu}$ and $\bar{\lambda}$ depend only upon the specific solution $W_\omega[h, f]$, the same is true for the uniquely defined κ .

Equations (87) and (88) then reduce to

$$\mu(M[h, f]) = \kappa(\bar{\lambda}[h, f]), \quad (92)$$

$$\lambda(M[h, f]) = \bar{\lambda}[h, f], \quad (93)$$

which admit the obvious solution

$$\mu(M) = \kappa(\lambda(M)). \quad (94)$$

This is the solution to the problem.

• An Application: As an example, consider the combination $G = h^2 - f$. Using equations (51) and (50), the unique expressions $\bar{\mu}$ and $\bar{\lambda}$ can be found as

$$\bar{\mu} = \frac{1}{2} \left(h - \frac{f}{h} \right) \quad \text{and} \quad \bar{\lambda} = \left(h - \frac{f}{h} \right). \quad (95)$$

The set of values of h and f for which G , $\bar{\mu}$ and $\bar{\lambda}$ are positive, real and negative respectively is given by the inequalities $h^2 > f$ and $h < 0$. For this range of values, the partial derivatives of G , $\bar{\mu}$, $\bar{\lambda}$ and \bar{R} are well defined, so the same applies to the whole procedure in chapters II-2 and II-3.

The unique real function κ is then determined by the relation

$$\kappa(\bar{\lambda}[h, f]) = \frac{1}{2} \bar{\lambda}[h, f] \quad (96)$$

and solves the problem. In particular, any negative-valued, real or complex, function $\lambda(M)$ is fine, provided that $\mu(M)$ satisfies the following equation,

$$\mu(M) = \kappa(\lambda(M)) = \frac{1}{2} \lambda(M). \quad (97)$$

Notice that this means that G can be derived from a variety of actions of the form (53).

Chapter II-5. The Algebra In Vacuum Gravity.

● The above procedure is part of a phenomenological approach towards the interpretation of the Brown-Kuchař algebra. The main conclusion is that the association of the combinations G and Λ_{\pm} with matter is a generic property of the solutions (39)-(40). Considering the diverse origins of the calculations in chapters II-2 and II-3, the coincidence of the corresponding results is remarkable, and suggests that the role of matter in general relativity deserves to be investigated further.

● The Time Evolution Generated By The Solutions: On the other hand, little progress has been made towards understanding the relevance of the solutions (39)-(40) in vacuum gravity. The necessity to stay away from the constraint surface of this theory, and possibly return to it when all calculations have been finished, is evident throughout the previous chapters. This is particularly clear for the combinations G and Λ_{\pm} since the time evolution they generate is, respectively, zero and ill-defined on the constraint surface of vacuum general relativity. The question that arises is whether this property applies for the whole family of solutions (39)-(40).

Consider a solution W that belongs to this family. The time evolution it generates when acting on an arbitrary local functional F of the gravitational canonical variables is the following,

$$\{F, W[h, f]'\} = \{F, h'\}W_h' + \{F, f'\}W_f'. \quad (98)$$

The primed quantities are evaluated at the spatial point x' .

A minimum prerequisite for the use of these W s in the vacuum theory is that they should create a constraint surface that is locally equivalent to the usual one. In particular, under the replacement of h by \mathcal{W} , the conditions $W \simeq 0$ and $f \simeq 0$ must imply the conditions $h \simeq 0$ and $f \simeq 0$, and vice versa. Notice that the constraint $f \simeq 0$ is equivalent to the usual constraint $\mathcal{H}^{gr}_i \simeq 0$ due to the positivity of the spatial metric. Equation (38) then implies that W_h must vanish weakly on the constraint surface of the vacuum theory,

$$W_h \simeq 0, \quad (99)$$

provided that W_h and W_f are both well-defined. No restriction is imposed on the value of W_f on the constraint surface.

When condition (99) is substituted in the evolution equation (98), the first term on the right side does not contribute because it vanishes weakly. However, the same is true for the second term assuming that W_f is well defined. This follows from the fact that f , defined by equation (37), is quadratic in the super-momenta \mathcal{H}^{gr}_i so that

$$\{F, f'\} = \{F, \frac{1}{g'} g^{ij'}\} \mathcal{H}^{gr}_{i'} \mathcal{H}^{gr}_{j'} + 2 \{F, \mathcal{H}^{gr}_{i'}\} \frac{1}{g'} g^{ij'} \mathcal{H}^{gr}_{j'} \simeq 0 \quad (100)$$

when the constraint $\mathcal{H}^{gr}_i \simeq 0$ is imposed. Therefore, the time evolution associated with any well-defined solution of equation (38) is trivial.

The inadequacy of the family of solutions (39-40) in vacuum gravity provides a further justification of the comment made in chapter II-2 concerning the exclusion of the trivial solutions. These solutions were excluded because they are not related to the action (53) and, also, because the time evolution they generate in the vacuum theory is trivial.

- Finding Vacuum Solution Of The Algebra: Although the solutions (39)-(40) are not regular in the sense of Dirac[11], the possibility that they are used in a quantum theory of vacuum gravity cannot be excluded. The combination G , for example, leads to a quadratic “Wheeler-DeWitt” equation when imposed as a restriction on the quantum states of the system, and no mention of its partial derivatives is required. Despite its quadratic character, the new equation could then result in an overall simplification in the quantum theory on account of its property of generating a genuine Lie algebra.

This possibility should be considered in more detail. Recall that the conditions $W_\omega \simeq 0$ and $h \simeq 0$ must imply each other when the constraint $f = 0$ is imposed. However, depending on the choice of sign for the square root in equation (42), the constraint $W_\omega \simeq 0$ may be satisfied identically when $f \simeq 0$. If this is the case it cannot enforce the necessary for equivalence Hamiltonian constraint. Conveniently, the sign-unambiguous expression for \bar{R} in equations (50)-(52) makes certain that this is not the case. This is because \bar{R} and $h - \bar{\mu}$ have the same sign when $f \simeq 0$, which means that W_ω in equation (42) does not become identically trivial.

Regardless of whether the solutions (39)-(40) can be used in a quantum theory of vacuum gravity, the issue that matters is whether alternative combinations that commute with themselves and lead to a well-defined dynamical evolution can exist in the vacuum theory. A further insight into the origin of the new algebra is required, and this can be attained through finding a geometric interpretation. Significant progress is made by comparing the new algebra with the Dirac one. The geometric interpretation of the latter is known, and the method for finding its physically relevant solutions is also available.

The geometry behind the Dirac algebra was recognized by Teitelboim[19], while the procedure for passing to its physical representations was developed by Hojman, Kuchař and Teitelboim in the derivation of geometrodynamics from first principles[20]. The attempt to adapt their method to the requirements of the new algebra results in a re-examination of the equal-time formalism and implies the need for “classical histories”. The issue of the interpretation of the new algebra is set aside for the moment, and is discussed again later, in part V.

Part III:

Introducing Classical Histories

Chapter III-1. Motivation.

● Description: The issue that is discussed in the following two parts concerns the transition from a general canonical Hamiltonian of the form $NH + N^i H_i$ to a specific canonical representation. The form of the Hamiltonian is general enough to incorporate a variety of canonical field theories, including general relativity. The information that distinguishes one theory from another comes solely through the choice of the canonical variables. The interpretation of the functions N and N^i is left arbitrary.

Some of this has been discussed already by Hojman, Kuchař and Teitelboim[20] who recovered geometrodynamics and other canonical representations of covariant theories from an algorithm involving a few plausible postulates. However, as stated by the authors themselves, the reduction of these postulates to the minimum was not attempted and some redundancy was left in the system. A few redundant requirements were pointed out at the end of their paper but, still, the exact relationship between the remaining postulates was not clarified and a further reduction seemed to be possible.

It is shown below that the complete set of postulates in [20] can be derived from the requirement that the canonical Hamiltonian is of the form $NH + N^i H_i$. The only other input that is needed in order that the most general canonical representation of H and H_i be found is the actual choice of the canonical variables. This choice involves some additional assumptions which are pointed out in a following section.

Besides, just as interesting as this result is the ensuing observation that the history formal-

ism seems to be superior to its standard (equal-time) counterpart. Whether this can be established as a general theorem, or not, depends on whether it is possible to find a direct link between the equal-time and the history approaches. What *is* established, however, is the fact that the spacetime meaning and connection of the postulates of Kuchař *et al* cannot be clarified in the standard formalism. The advantages of the history formalism over the standard approach are therefore genuine, at least when discussing *spacetime* issues. This is enough to suggest a history approach to quantum gravity.

To be more precise, the original aim of Kuchař *et al* was to use postulates that depend exclusively on the three-dimensional hyper-surface. It is only for this reason that the relationship between their individual postulates is not clear. If the assumption of a surrounding spacetime (which is already implicit in their arguments) is used explicitly as an additional postulate then the meaning and connection of all the remaining postulates is clarified. However, the clarification of the postulates cannot be achieved in a formalism based on equal-time Poisson brackets.

This is because, in an equal-time formalism, Poisson brackets that involve the time derivatives of the canonical variables cannot be defined; at least, not without the addition of further structure. Seen from a spacetime perspective, however, these brackets ought to be treated in an equivalent way, in which case they would give additional information about the theory's kinematics. In the equal-time formalism the missing information is recovered precisely by the additional postulates imposed in [20]; most notably that of the Dirac algebra. On the other hand, the present approach is based on a Hamiltonian formalism whose phase space includes the fields at general times; i.e., is defined over the space of

classical histories. The correspondence with the spacetime picture is therefore exact and the reduction of the postulates comes as a direct consequence.

The discussion in this section is organized as follows. In the remaining of chapter III-1 the existing work on the subject is reviewed, and the main issues are emphasized. In chapter III-2 the Hamiltonian formalism defined over the space of classical histories is introduced. It incorporates both constrained and unconstrained systems and, at least for the issues of interest, is a simpler alternative to the Dirac method. In chapter III-3 the history formalism is employed to transform the postulated form $NH + N^i H_i$ of the canonical Hamiltonian to a set of kinematic conditions on the canonical generators. These conditions define the *evolution* postulate which, together with the additional assumption of a surrounding spacetime, is then used in part IV to derive the canonical representations.

- **The Dirac Algebra And The Principle Of Path Independence:** The super-Hamiltonian and the super-momentum of vacuum general relativity are not the only canonical generators that close according to the Dirac algebra. The latter is satisfied by the canonical generators of a parametrized field theory and, in a modified form, by the generators of any field theory that is not parametrized.

Its universality implies that the Dirac algebra is connected with a geometric property of spacetime that is independent of the specific dynamics of the canonical theory. The fact that the Dirac algebra is a kinematic consistency condition was shown by Teitelboim[19] who derived it from a geometric argument corresponding to the integrability of Hamilton's equations. This consistency argument—termed by Kuchař[29] “the principle of path independence of the dynamical evolution”—ensures that the change in the canonical variables

during the evolution from a given initial surface to a given final surface is independent of the particular sequence of intermediate surfaces used in the actual evaluation of this change.

Besides the assumption of path independence—which applies regardless of the specific form of the canonical Hamiltonian—Teitelboim’s proof also involved explicitly the assumption that the Hamiltonian is decomposed according to the lapse-shift formula written in equation (9). Using these two postulates, together, he concluded that in order for the theory to be consistent the phase space should be restricted by the initial value equations (10-11) while the canonical generators should satisfy the Dirac algebra (12-14).

The very last statement is not completely true because of a mistake in the reasoning in [19] concerning the fact that the system is constrained. Nevertheless, the correct algebra—as it arises from the requirement of path independence—is still the Dirac one, but is supplemented by terms G , G_i and G_{ij} whose first partial derivatives with respect to the canonical variables vanish on the constraint surface:

$$\{H(x), H(x')\} = g^{ij}(x)H_i(x)\delta_{,j}(x, x') + G(x, x') - (x \leftrightarrow x'), \quad (101)$$

$$\{H(x), H_i(x')\} = H(x)\delta_{,i}(x, x') + H_{,i}(x)\delta(x, x') + G_i(x, x'), \quad (102)$$

$$\{H_i(x), H_j(x')\} = H_j(x)\delta_{,i}(x, x') + G_{ij}(x, x') - (ix \leftrightarrow jx'). \quad (103)$$

The derivation of the above set of relations, which are called the “weak Dirac algebra”, can be found in part IV.

● The Problem Of Deriving A Physical Theory From Just The Canonical Algebra: The principle of path independence was an indication that a Hamiltonian theory does not

have to depend exclusively on the canonical decomposition of a given spacetime action but may have also an independent status. However, in any attempt to construct specific canonical theories via the principle of path independence, alone, some information is found to be missing. The weak Dirac algebra admits numerous representations whose physical relevance is therefore doubtful.

As an example, consider the case when the canonical variables are the spatial metric and its conjugate momentum. The usual strong limit of the algebra (101-103) is taken, where all the terms G , G_i and G_{ij} are identically zero. The generator H_i is chosen as the super-momentum of the gravitational field,

$$H_i(x) = \mathcal{H}^{gr}_i(x), \quad (104)$$

and the normal generator H is required to transform as a scalar density of weight one. Under these simplifications, the second and third Dirac relations (13-14) are satisfied, and the Dirac algebra—which can be seen as a set of coupled differential equations for the canonical generators—decouples. This leaves a single first-order equation for $H(x)$, equation (12), which is expected normally to admit an infinite number of distinct solutions. In particular, it can be assumed that $H(x)$ is of the form[16]

$$H(x) = g^{\frac{1}{2}} W[h, f](x). \quad (105)$$

The weight-zero scalar densities h and f are the ones defined in [16] as well as in equations (36)-(37). When the ansatz (105) is used in equation (12), a differential equation for the function $W[h, f]$ arises,

$$\frac{1}{2} W W_f = f W_f^2 - \frac{1}{4} W_h^2 + \frac{1}{4}. \quad (106)$$

In analogy with the ω -equation (38), it admits a family of solutions parametrized by an arbitrary function of one variable.

The general solution of equation (106) is obtained by solving in terms of W the complete integral

$$\left(W + \sqrt{W^2 - 4f + 4[h - B(a[h, f])]^2} \right) \times \\ \times \exp\left(\frac{W}{W + \sqrt{W^2 - 4f + 4[h - B(a[h, f])]^2}} \right) + a[h, f] = 0. \quad (107)$$

As usual in these cases[28], the form of the function $a[h, f]$ is determined by solving algebraically in terms of a the equation

$$\frac{\partial}{\partial a} \left\{ \left(W + \sqrt{W^2 - 4f + 4[h - B(a)]^2} \right) \times \right. \\ \left. \times \exp\left(\frac{W}{W + \sqrt{W^2 - 4f + 4[h - B(a)]^2}} \right) + a \right\} = 0 \quad (108)$$

for a given choice of the function $B(a)$.

The super-Hamiltonian of general relativity, arising when $W[h, f] = h$, is the only one of these solutions that is ultra-local in the field momenta. The ultra-locality is actually related to the geometric meaning of the canonical variables but this will be discussed in detail later, in part IV. For the moment notice that if the weak Dirac algebra (101-103) is used as the starting point of the above calculation—which is the correct thing to be done—then a set of differential equations arises whose precise form is not known and, therefore, no further progress can be made.

● **Selecting The Physical Representations Of The Dirac Algebra:** Deriving geometrodynamics from plausible first principles[20], Hojman, Kuchař and Teitelboim chose to lay the

stress on the concept of infinite dimensional groups and placed the strong Dirac algebra at the centre of their approach. They expected that the closing relations (12-14) carry enough information about the system to select a physical representation uniquely but they could not extract this information directly from them. The existence of solutions like (105) is the reason why.

What the authors of [20] did, instead, was to follow an indirect route and select the physically relevant representations by supplementing the strong Dirac algebra with four additional conditions. Specifically, they introduced the tangential and normal generators of hyper-surface deformations, defined respectively by

$$H^D{}_i(x) := \mathcal{X}^\alpha{}_i(x) \frac{\delta}{\delta \mathcal{X}^\alpha}(x), \quad (109)$$

$$H^D(x) := n^\alpha(x) \frac{\delta}{\delta \mathcal{X}^\alpha}(x), \quad (110)$$

and acted with these on the spatial metric:

$$H^D{}_k(x') g_{ij}(x) = g_{ki}(x) \delta_{,j}(x, x') + g_{kj}(x) \delta_{,i}(x, x') + g_{ij,k}(x) \delta(x, x'), \quad (111)$$

$$H^D(x') g_{ij}(x) = 2n_{\alpha;\beta}(x) \mathcal{X}^\alpha{}_i(x) \mathcal{X}^\beta{}_j(x) \delta(x, x'). \quad (112)$$

They required that equations (111-112)—which are purely kinematic and hold in an arbitrary Riemannian spacetime—should be satisfied by the canonical generators,

$$\{g_{ij}(x), H_k(x')\} = g_{ki}(x) \delta_{,j}(x, x') + g_{kj}(x) \delta_{,i}(x, x') + g_{ij,k}(x) \delta(x, x'), \quad (113)$$

$$\{g_{ij}(x), H(x')\} \propto \delta(x, x'), \quad (114)$$

so that any dynamics in spacetime would arise as a different canonical representation of the universal kinematics. Notice that only the ultra-locality of the second Poisson bracket

was used. The justification and geometric interpretation of equations (113) and (114) can be found in [20].

The strong Dirac algebra with the conditions (113) and (114) results in a unique representation for the generators H and H_i . It corresponds to the super-Hamiltonian (10) and the super-momentum (11) of general relativity. The requirement of path independence is then imposed as an additional postulate to the algebra. It enforces the initial value constraints (10-11) and, therefore, the complete set of Einstein's equations is recovered. The most general scalar field Lagrangian with a non-derivative coupling to the metric can be derived along similar lines[29].

•The Full List Of The Selection Postulates: The precise assumptions used by the authors are summarized at the end of their paper. They are written here in an equivalent form and, in the case of pure gravity, they are the following:

(i) The evolution postulate: The metric and its conjugate momentum are regarded as the sole canonical variables. There exists a Hamiltonian that generates the dynamical evolution of the theory. It can be casted in the lapse-shift form, equation (9), where the super-Hamiltonian and super-momentum generators are constructed entirely from the canonical variables.

(ii) The representation postulate: The canonical generators must satisfy the closing relations (12-14) of the strong Dirac algebra.

(iii) Initial data re-shuffling: The Poisson bracket (113) between the super-momentum and the configuration variable g_{ij} must coincide with the kinematic relation (111).

(iv) Ultra-locality: The Poisson bracket (114) between the super-Hamiltonian and the

configuration variable g_{ij} must coincide with the kinematic relation (112).

(v) Reversibility: The time-reversed spacetime must be generated by the same super-Hamiltonian and super-momentum as the original spacetime.

(vi) Path independence: The dynamical evolution predicted by the theory must be such that the change in the canonical variables during the evolution from a given initial surface to a given final one is independent of the actual sequence of intermediate surfaces used in the evaluation of this change.

Notice that there is an implicit assumption hidden in the evolution postulate. Namely, in order that the metric and the momentum be a canonical pair, the metric should be a spatial scalar and the momentum a spatial density of weight one. This is necessary since, otherwise, the δ -function appearing in the basic Poisson bracket relations does not have the appropriate spatial weight. Specifically, it should be a scalar in the first argument and a density in the second. Notice, also, that in all the other postulates an assumption of a surrounding spacetime is implied. Later, this assumption will be identified explicitly, as an additional postulate concerning the choice of canonical variables. The postulates (ii)-(vi) will then be shown to be unnecessary.

•The Need For a Detailed Understanding Of The Selection Postulates: The above assumptions comprise a set of plausible first principles on which the canonical formulation of a theory can be based. However, in a sense these principles are not completely satisfying. This is because they do not correspond to a minimum set and because the connection between them is not clear. The authors of [20] mentioned the redundancy of the reversibility postulate (v) as well as the fact that the third closing relation of the representation postu-

late (ii) is made redundant by the re-shuffling requirement (iii). They stressed the need for understanding the precise reason why some equations hold strongly while others hold only weakly and, in particular, for clarifying the relationship between the strong representation postulate (ii) and the weak requirement of path independence (vi).

The revised form of Teitelboim's argument makes such a clarification a more important issue since, now, the strong representation requirement—which is at the heart of the approach in [20]—seems to be unjustified. In addition, repeating the geometric argument used in [19] in the reverse order, it follows that the dynamical evolution of the theory must also hold weakly. This is in contrast to the strong equations used in postulates (iii) and (iv). On the other hand, recall that any attempt to replace these equations by weak ones will result in a situation where the particular form of the differential equations that need to be solved will not be known and no further progress will be made.

Putting the issue of the weak equalities aside, the understanding of the exact relationship between the postulates is needed if the method in [20] is to be applied to the case of a generic canonical algebra. The reason is that—in the existing formulation of the postulates—the overall consistency is made certain only by the fact that the re-shuffling and ultra-locality assumptions (iii) and (iv) are respected by the dynamical law of the theory (i). On the other hand, the remaining postulates do not ensure that assumptions (iii) and (iv) are the only ones compatible with this law. If different compatible assumptions are used as supplementary conditions to the algebra then the method in [20] will yield different canonical representations.

However, the dynamical law of the theory is the only assumption that enters the derivation

and geometric interpretation of the algebra besides the principle of path independence. It follows that the existing formulation of the postulates will be ambiguous if it is used as an algorithm for passing from the interpretation of an algebra to its physical representations. This is of course particularly relevant to the discussion made in part II concerning the interpretation of the new Lie algebra.

Finally, there is an asymmetry in the formulation of the postulates within which lies the main motivation for the discussion in this part. It concerns the identification of the canonical generators with the generators of normal and tangential hyper-surface deformations, that is required to hold in postulates (iii) and (iv) for the configuration variable only. However, if such an identification is a fundamental principle in the canonical theory then it should hold for both the canonical variables, in which case additional information about the kinematics of the system can be extracted.

In an equal-time formalism, this conjecture can neither be confirmed nor rejected because the action of the deformation generators on the canonical momenta cannot be defined. Marolf[30] used the Hamiltonian as an additional structure to extend the Poisson bracket from a Lie bracket on phase space to a Lie bracket on the space of histories. Here, instead, the equal-time formalism is put aside, and a phase space is introduced whose Poisson bracket is defined over the space of histories from the beginning.

Chapter III-2. The History Phase Space.

In this chapter, the history formulation of canonical dynamics is presented. Since most of the results are merely translated from the standard approach, the discussion is rather basic. For example, no detailed analysis is given of the Dirac method for dealing with constraints, or with the gauge transformations they generate. Only those features of the history formalism are given that are needed for the discussion that follows. The precise connection between the present section and the previous chapters will become apparent in chapter III-3, where the evolution postulate is re-formulated in terms of classical histories.

- The Unconstrained Hamiltonian: Consider the theory described by the canonical action

$$\begin{aligned} S[q^A, p_A] &= \int d^3x dt \left(p_A \dot{q}^A - \mathcal{H} \right), \\ \mathcal{H} &= NH + N^i H_i. \end{aligned} \tag{115}$$

The functions N and N^i are fixed (i.e., non-canonical) functions of space and time. The generators H and H_i are functions of the canonical fields q^A , p_A and may also depend on additional fixed fields c^K . The index A runs from 1 to half the total number of canonical variables, while K runs from 1 to the total number of fixed fields.

The phase space can be generalized to include the canonical fields at all times. This can be done by introducing the space of histories,

$$\left(q^A(x, t), p_A(x, t) \right), \tag{116}$$

and defining on it the Poisson bracket

$$\{q^A(x, t), p_B(x', t')\} = \delta^A_B \delta(x, x') \delta(t, t'). \quad (117)$$

The quantum analogue of the canonical fields in (116) is the one-parameter family of Schrödinger operators introduced by Isham *et al* in their study of continuous time consistent histories[3,4].

The Poisson bracket (117) turns the space of histories into a Poisson manifold. In terms of this bracket, the variation of the canonical action can be written concisely in the form

$$\{S, q^A(x, t)\} \simeq 0, \quad (118)$$

$$\{S, p_A(x, t)\} \simeq 0 \quad (119)$$

and defines a constraint surface on the space of histories. The physical fields are defined to satisfy these relations for each value of x and t . For the particular form (115) of the canonical action, the weak equations (118-119) become¹

$$\dot{q}^A(x, t) \simeq \int d^3x' dt' \{q^A(x, t), \mathcal{H}(x', t')\} \equiv \int d^3x' \frac{\delta \mathcal{H}}{\delta p_A}(x', t) \delta(x, x') \quad (120)$$

$$\dot{p}_A(x, t) \simeq \int d^3x' dt' \{p_A(x, t), \mathcal{H}(x', t')\} \equiv \int d^3x' \frac{\delta \mathcal{H}}{\delta q^A}(x', t) \delta(x, x'), \quad (121)$$

which can be recognized as Hamilton's equations in the usual equal-time sense. This follows from the fact that the Hamiltonian in equation (115) is by construction independent of any time derivatives, so it can be integrated trivially over $\int dt' \delta(t, t')$.

¹Throughout the thesis, the functional derivative $\frac{\delta \mathcal{F}}{\delta q^A}$ is defined by $\frac{\delta \mathcal{F}}{\delta q^A} = \frac{\partial \mathcal{F}}{\partial q^A} + \frac{\partial \mathcal{F}}{\partial q^A_{,i}} \partial_i + \frac{\partial \mathcal{F}}{\partial q^A_{,ij}} \partial_{ij} + \dots etc$. Sometimes \mathcal{F} is called a functional, although it is only a local function of the canonical variables and a finite number of their derivatives.

The weak equality sign is a reminder of the fact that Hamilton's equations, and consequently the theory, are not preserved under a general Poisson bracket. In the equal-time formalism this presents no problem because the canonical velocities are only defined externally but, here, they are included equally in the phase space. For example, the Poisson bracket between a field velocity and its conjugate momentum can be evaluated to give a time derivative of the δ -function. This is not the result that arises when the corresponding Hamilton equation is used to replace the field velocity before the commutation is performed. Nonetheless, since the theory is about time evolution only, it is sufficient that Hamilton's equations are preserved weakly under the Poisson bracket with the Hamiltonian.

In the unconstrained theory (115) this follows automatically from Hamilton's equations and the definition of the history Poisson bracket (117) without any reference to the specific form of the Hamiltonian. However, before this can be checked directly, the definition of the Hamiltonian has to be extended so that it can incorporate the trivial dynamical evolution of the fixed functions c^K , N and N^i . This is also appropriate for the completeness of the formalism.

- Incorporating The Fixed Functions: The extended unconstrained action is defined by

$$S[q^A, p_A, \omega_K, \omega, \omega_i] = \int d^3x dt \left(p_A \dot{q}^A + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i - \mathcal{H}^{ext} \right),$$

$$\mathcal{H}^{ext} = NH + N^i H_i + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i. \quad (122)$$

The momenta ω_K , ω and ω_i are defined through the Poisson bracket relations

$$\{c^K(x, t), \omega_L(x', t')\} = \delta^K_L \delta(x, x') \delta(t, t'),$$

$$\{N(x, t), \omega(x', t')\} = \delta(x, x') \delta(t, t'),$$

$$\{N^i(x, t), \omega_j(x', t')\} = \delta^i_j \delta(x, x') \delta(t, t'). \quad (123)$$

The various δ -functions transform in different ways depending on the transformation properties of the corresponding canonical variables. This is not denoted explicitly in order to keep the notation simple. Furthermore, the above momenta are not assumed to have any direct physical significance or interpretation. The whole purpose of their introduction is to allow the time derivative of the fixed functions to be calculated inside the Poisson bracket formalism.

Restricting the discussion to functionals of the canonical and the fixed variables, it follows that

$$\begin{aligned} \{F(x, t), \int d^3x' dt' \mathcal{H}^{ext}(x', t')\} &= \frac{\delta F}{\delta q^A}(x, t) \{q^A(x, t), \int d^3x' dt' \mathcal{H}^{ext}(x', t')\} \\ &+ \frac{\delta F}{\delta p_A}(x, t) \{p_A(x, t), \int d^3x' dt' \mathcal{H}^{ext}(x', t')\} + \frac{\delta F}{\delta c^K}(x, t) \dot{c}^K(x, t) \\ &+ \frac{\delta F}{\delta N}(x, t) \dot{N}(x, t) + \frac{\delta F}{\delta N^i}(x, t) \dot{N}^i(x, t) \simeq \dot{F}(x, t). \end{aligned} \quad (124)$$

This implies that the extended Hamiltonian is the canonical representation of the total time derivative operator.

Equivalently it may be observed that when the kinematic half of the extended action,

$$\int d^3x dt \left(p_A \dot{q}^A + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i \right), \quad (125)$$

is acting on the functional F it produces the time derivative of F in the strong sense. On the other hand, when the total extended action acts on any F it yields weakly zero by definition.

It follows that the remaining half of the action—i.e., the dynamical half corresponding to the integral of the extended Hamiltonian—produces the total time derivative of F in the

weak sense.

The above result implies that Hamilton's equations are preserved automatically under the dynamical evolution of the theory. Indeed, if F is any functional of the canonical and the fixed variables that vanishes on the constraint surface, its total time derivative also vanishes on the same surface. Since this derivative is weakly equal to the commutation of F with the integral of the extended Hamiltonian, it follows that all weakly vanishing F s remain weakly zero under this commutation. Choosing these F s to be Hamilton's equations themselves shows that the constraint surface is preserved. This completes the treatment of systems that are unconstrained in the usual sense.

● The Constrained Hamiltonian: The extended form of the action, equation (122), arises naturally when the functions N and N^i are either constrained canonical variables or acquire the meaning of Lagrange multipliers. Both these cases are presented in their most general form by considering the canonical action

$$S[q^A, p_A, N, \omega, N^i, \omega_i, \omega_K] = \int d^3x dt \left(p_A \dot{q}^A + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i - \mathcal{H} \right),$$

$$\mathcal{H} = NH + N^i H_i + \omega_K \dot{c}^K + \omega \dot{N} + \omega_i \dot{N}^i, \quad (126)$$

which is now varied additionally with respect to N and N^i . The fields c^K are still treated as fixed.

The variation of (126) leads to the same equations as before, namely

$$\{S, q^A(x, t)\} \simeq 0 \Leftrightarrow \dot{q}^A(x, t) \simeq \int d^3x' \left(N \frac{\delta H}{\delta p_A} + N^i \frac{\delta H_i}{\delta p_A} \right)(x', t) \delta(x, x'), \quad (127)$$

$$\{S, p_A(x, t)\} \simeq 0 \Leftrightarrow \dot{p}_A(x, t) \simeq \int d^3x' \left(N \frac{\delta H}{\delta q^A} + N^i \frac{\delta H_i}{\delta q^A} \right)(x', t) \delta(x, x'), \quad (128)$$

$$\{S, c^K(x, t)\} = 0 \Leftrightarrow \dot{c}^K(x, t) = \dot{c}^K(x, t) \Leftrightarrow 0 = 0, \quad (129)$$

$$\{S, N(x, t)\} = 0 \Leftrightarrow \dot{N}(x, t) = \dot{N}(x, t) \Leftrightarrow 0 = 0, \quad (130)$$

$$\{S, N^i(x, t)\} = 0 \Leftrightarrow \dot{N}^i(x, t) = \dot{N}^i(x, t) \Leftrightarrow 0 = 0, \quad (131)$$

subject to the additional equations

$$\{S, \omega(x, t)\} \simeq 0 \Leftrightarrow \dot{\omega}(x, t) \simeq \dot{\omega}(x, t) + H(x, t) \Leftrightarrow H(x, t) \simeq 0, \quad (132)$$

$$\{S, \omega_i(x, t)\} \simeq 0 \Leftrightarrow \dot{\omega}_i(x, t) \simeq \dot{\omega}_i(x, t) + H_i(x, t) \Leftrightarrow H_i(x, t) \simeq 0 \quad (133)$$

arising from the variation of the action with respect to N and N^i .

For a functional $F[q^A, p_A, c^K, N, N^i]$ the proof of the previous section still applies,

$$\{F(x, t), \int d^3x' dt' \mathcal{H}(x', t')\} \simeq \dot{F}(x, t), \quad (134)$$

with the weak equality referring to Hamilton's equations (127-128). Again, if F is any functional that vanishes on the surface defined by Hamilton's equations, its time derivative also vanishes on this surface. Therefore, by taking Hamilton's equations to be these F s, it can be deduced that equations (127-131) are preserved weakly under the dynamical evolution of the theory. On the other hand, if F vanishes on the surface defined by the constraint equations (132-133), its time derivative still vanishes on the this surface but, now, it does not follow that this is the time derivative generated by the Hamiltonian of the theory.

It must be ensured also that the time derivatives of the fields evaluated by differentiating equations (132-133) are compatible with the time derivatives of the same fields evaluated from Hamilton's equations. If the constraints (132-133) do not depend on the fixed fields c^K —which is the case for most of the physical theories—this compatibility condition results

in the requirement that the algebra of H and H_i must close weakly under the history Poisson bracket. Since H and H_i are by construction independent of any time derivatives the weak closure of the algebra refers only to the constraint equations (132-133).

Chapter III-3. The Evolution Postulate.

- The Inverse Procedure And The Evolution Postulate: The aim is to invert the above argument, and recover the general canonical Hamiltonian of a theory from a set of first principles. The need that these principles be minimal implies that the appropriate starting point of the derivation is the requirement that the canonical action should have the form (126). This conclusion follows from the observation that equation (126) is the only prerequisite for the existence of a canonical algebra in the theory.

According to the terminology used in [20], equation (126) corresponds to the “evolution postulate”. In case that this postulate turns out to be insufficient to determine the theory

completely, the plan is that any supplementary conditions that may be added must be such that the connection between them remains clear throughout the derivation.

Initially, the most general canonical representation of the Hamiltonian is sought that satisfies the unconstrained version of the postulate,

$$\frac{\partial}{\partial t} q^A(x, t) \simeq \int d^3 x' dt' \{q^A(x, t), (NH + N^i H_i)(x', t')\}, \quad (135)$$

$$\frac{\partial}{\partial t} p_A(x, t) \simeq \int d^3 x' dt' \{p_A(x, t), (NH + N^i H_i)(x', t')\}. \quad (136)$$

If such a Hamiltonian cannot be found, there still is the alternative possibility of varying the action with respect to the functions N and N^i . Equations (135-136) must then be supplemented by the constraint equations

$$H(x, t) \simeq 0, \quad (137)$$

$$H_i(x, t) \simeq 0, \quad (138)$$

that have to be preserved under the dynamical evolution of the theory. This consistency requirement is included in the evolution postulate for constrained systems, and amounts to the weak closure of the algebra when the constraints are independent of fixed fields.

Notice that the time derivatives of N and N^i do not appear in the equations of motion (135-138) which means that N and N^i are allowed to have arbitrary numerical values. This is also true, by definition, for the non-dynamical N and N^i in the case of unconstrained systems. Therefore, the evolution postulate for both constrained and unconstrained systems can be re-stated as the requirement that the canonical action should be of the form (126) with N and N^i taking arbitrary numerical values. In practice, the lapse function is still required to be positive.

● The Evolution Postulate In An Equivalent Form: At first sight, conditions (135-136) do not seem to be restrictive enough so that something definite can be drawn from them. It seems that the canonical representations can be chosen at will, and that any constrained theory can be created by just requiring the closure of the resulting algebra. This view changes when the precise geometric meaning of the canonical fields is taken into account. For example, in a scalar field theory, the field $\phi(x, t)$ is not merely a spatial scalar but is also by definition the pull-back of a spacetime scalar field. Below, the evolution postulate is transformed to an equivalent condition on the canonical generators that is more appropriate for the exploitation of this fact.

The functions N and N^i are taken as the lapse function and the shift vector. In fact, this will be the case henceforth unless stated otherwise. If the time derivative operator in equations (135-136) is decomposed according to the lapse-shift formula,

$$\frac{\partial}{\partial t} = N n^\alpha \frac{\partial}{\partial X^\alpha} + N^i \mathcal{X}^\alpha_i \frac{\partial}{\partial X^\alpha}, \quad (139)$$

and the momentum \mathcal{P}_α conjugate to the embedding is introduced,

$$\{\mathcal{X}^\alpha(x, t), \mathcal{P}_\beta(x', t')\} = \delta^\alpha_\beta \delta(x, x') \delta(t, t'), \quad (140)$$

equation (135-136) takes the following form,

$$\{q^A(x, t), H(x', t')\} \simeq \{q^A[\mathcal{X}](x, t), \mathcal{P}_\beta(x', t')\} n^\beta(x', t'), \quad (141)$$

$$\{q^A(x, t), H_i(x', t')\} \simeq \{q^A[\mathcal{X}](x, t), \mathcal{P}_\beta(x', t')\} \mathcal{X}^\beta_i(x', t'), \quad (142)$$

$$\{p_A(x, t), H(x', t')\} \simeq \{p_A[\mathcal{X}](x, t), \mathcal{P}_\beta(x', t')\} n^\beta(x', t'), \quad (143)$$

$$\{p_A(x, t), H_i(x', t')\} \simeq \{p_A[\mathcal{X}](x, t), \mathcal{P}_\beta(x', t')\} \mathcal{X}^\beta_i(x', t'). \quad (144)$$

Notice that the arbitrariness of N and N^i has been used to eliminate the integration.

On the right side of the above equations the explicit dependence of the canonical fields on the spacetime embedding is taken into account. For the configuration fields this is just the dependence arising from the definition of the fields as geometric objects in spacetime. For the conjugate fields the situation is more complicated, and equation (135) is assumed to have been inverted to express the momenta as functionals of the configuration variables, the lapse, the shift, and the prescribed fields c^K . All the latter have a definite dependence on the spacetime embedding which is then conveyed to the conjugate canonical fields.

Equation (135) is always invertible for the momenta because, by construction, the system is constrained only in the quantities N and N^i at the most. There is one exception to this rule when the action is not derivable from a spacetime Lagrangian but, instead, is brought into the form (126) through the introduction of Lagrange multipliers. This is the relevant case for parametrized theories.

For the purposes of performing actual calculations, the evolution postulate is to be used in the following way. Any time derivatives of the canonical variables that arise on the right side of equations (141-144) are replaced by the original Hamilton's equations (135-136). When the theory is unconstrained, this results in a coupled system of four functional differential equations for H and H_i . If a solution exists, it corresponds to the general canonical representation compatible with the evolution postulate. On the other hand, when the theory is constrained the resulting "equations" for H and H_i are not proper differential equations, since it is sufficient that they hold only on the constraint surface (137-138).

If the constraints (137-138) implied that the canonical variables can not be treated as independent in these “equations” for H and H_i , the evolution postulate for constrained systems would not make any sense at all. However, by construction of the canonical formalism, the constraints must be imposed only *after* the Poisson brackets have been evaluated. Therefore, even for constrained systems, the differential equations for H and H_i should be solved as if the canonical variables were independent, and the constraints (137-138) should be imposed only at the end. In practice, a term is added on each differential equation, the value of which is required to vanish on the constraint surface. Exactly how this is done is shown in part IV, in the case of general relativity.

Finally, notice that the replacement of the field velocities in equations (141-144) with the original and equivalent equations (135-136) does not lead to cyclic identities as it might have been expected. The reason is that the original equations hold in integrated form, while equations (141-144) hold at every point in space and time due to the arbitrariness of N and N^i . The information incorporated in these equations is actually so rich that it determines the canonical representations of the theory.

Part IV:

Canonical Theories Derived From First Principles

Chapter IV-1. The Indirect Method Applied To Gravity.

● The New Set Of Postulates: It will be shown that the only assumptions needed for the derivation of the representations of a canonical theory are (a) the evolution postulate and (b) the explicit assumption of a surrounding spacetime. That this is indeed so will be shown in an indirect way, by starting from the above two assumptions and recovering the complete set of postulates of Kuchař *et al.* For constrained systems, it turns out that these postulates have to be imposed weakly, which is also implied by the revised version of Teitelboim's argument.

The new postulates are the following:

(a) The evolution postulate: The configuration variable (here the metric) and its conjugate momentum are regarded as the sole canonical variables. There exists a Hamiltonian that generates the dynamical evolution of the theory. It can be casted in the lapse-shift form, equation (9), where the super-Hamiltonian and super-momentum generators are constructed entirely from the canonical variables.

(b) The postulate of a surrounding spacetime: The canonical configuration variable is the pull-back of a configuration variable in spacetime through the foliation associated with the lapse function and shift vector appearing in the Hamiltonian.

For simplicity, both these postulates together will be called the evolution postulate in the chapters that follow.

● Recovery Of The Re-shuffling And Ultra-locality Postulates: On the right side of equa-

tions (141-142) the configuration fields are treated as functionals of the embedding relative to which the decomposition of the spacetime theory has been performed. The re-shuffling and ultra-locality postulates follow immediately from equations (141-142) once the geometric meaning of the configuration variable is taken into account (i.e., postulate (b) in the above notation). This is recognized in [20], although the emphasis is given on the compatibility of the postulates with the dynamical law (135-136) rather than on the fact that the postulates are determined by this law uniquely. Referring to the corresponding comment in chapter III-1, it is only because of this fact that the method in [20] can be used unambiguously as an algorithm for finding the physically relevant representations of a generic canonical algebra.

The ultra-locality and re-shuffling conditions are written down below for the physical examples that are usually considered. The relevant calculations can be found in Appendix C. Notice that a strong equality sign is used, with the understanding that all canonical velocities have been eliminated through the corresponding Hamilton's equations. This is consistent with the general plan, according to which an unconstrained representation of the evolution postulate is sought originally. If a theory is proved to be constrained the following equations will be revised accordingly. The presence of these canonical velocities in postulates (a) and (b) is of course the reason why none of the following arguments can be applied in the standard canonical formalism.

(i) Scalar field theory: The configuration variable is the pullback of a spacetime scalar field,

$$\phi(x, t) = \phi[\mathcal{X}](x, t), \tag{145}$$

and, as such, is an ultra-local function of the embedding. Equations (141-142) become

$$\{\phi(x, t), H_i(x', t')\} = \phi_{,\beta}(x, t) n^\beta(x, t) \delta(x, x') \delta(t, t') \quad (146)$$

$$\{\phi(x, t), H(x', t')\} = \phi_{,i}(x, t) \delta(x, x') \delta(t, t'), \quad (147)$$

which can be recognized as the history analogues of the re-shuffling and ultra-locality conditions in [20]. Indeed, the $\delta(t, t')$ function indicates that the canonical generators are independent of the field velocities, the ultra-locality of the first equation implies that the super-Hamiltonian is an ultra-local function of the momenta, while the form of the second equation ensures that the super-momentum just re-shuffles the data on the hyper-surface.

(ii) General relativity: The configuration variable is the pullback of the spacetime metric,

$$g_{ij}(x, t) = \gamma_{\alpha\beta}[\mathcal{X}](x, t) \mathcal{X}^\alpha{}_i(x, t) \mathcal{X}^\beta{}_j(x, t). \quad (148)$$

and equations (141-142) result in the following conditions on the canonical generators,

$$\begin{aligned} \{g_{ij}(x, t), H_k(x', t')\} &= g_{ki}(x, t) \delta_{,j}(x, x') \delta(t, t') + g_{kj}(x, t) \delta_{,i}(x, x') \delta(t, t') \\ &+ g_{ij,k}(x, t) \delta(x, x') \delta(t, t'), \end{aligned} \quad (149)$$

$$\{g_{ij}(x, t), H(x', t')\} = 2n_{\alpha;\beta}(x, t) \mathcal{X}^\alpha{}_i(x, t) \mathcal{X}^\beta{}_j(x, t) \delta(x, x') \delta(t, t'). \quad (150)$$

For the same reasons as in case (i) above, these can be recognized as the history analogues of the re-shuffling and ultra-locality postulates (113-114).

(iii) Deformation and parametrized theories: For the theory of hyper-surface deformations, the configuration variable is the embedding itself. Equations (141-142) become

$$\{\mathcal{X}^\alpha(x, t), H(x', t')\} = n^\alpha(x, t) \delta(x, x') \delta(t, t'), \quad (151)$$

$$\{\mathcal{X}^\alpha(x, t), H_i(x', t')\} = \mathcal{X}^\alpha_i \delta(x, x') \delta(t, t'), \quad (152)$$

which are the re-shuffling and ultra-locality conditions for the deformation theory. Using the equations in (i) and in (iii) together, the corresponding conditions for a parametrized scalar field theory arise.

● The Two Jacobi Identities: This is the revised version of the geometric argument in [19], so many of the following results can be found in [19] and [20]. They are re-stated here only for completeness. Besides the revision of the argument for constrained systems, the other difference between this approach and the approach in [20] is that the present discussion does not rely on the principle of path independence. The later is also derived from the evolution postulate.

The discussion starts from the following two Jacobi identities,

$$\begin{aligned} & \{\{H_j(x', t'), F(x'', t'')\}, H_i(x, t)\} + \{\{F(x'', t''), H_i(x, t)\}, H_j(x', t')\} \\ & + \{\{H_i(x, t), H_j(x', t')\}, F(x'', t'')\} = 0, \end{aligned} \quad (153)$$

$$\begin{aligned} & \{\{H^D_j(x', t'), F(x'', t'')\}, H^D_i(x, t)\} + \{\{F(x'', t''), H^D_i(x, t)\}, H^D_j(x', t')\} \\ & + \{\{H^D_i(x, t), H^D_j(x', t')\}, F(x'', t'')\} = 0, \end{aligned} \quad (154)$$

that hold on the canonical and on the deformation history phase space respectively. The arbitrary functional F depends on both the canonical variables q^A and p_A , while the action of the deformation generators on these variables is defined as in chapter III-3. The notation for the normal and tangential projections of P_α is chosen to coincide with the equal-time definitions (109-110).

The only case that is considered is when the canonical Hamiltonian is independent of the

fixed fields c^K , which is the relevant case for general relativity. When prescribed fields are present in the Hamiltonian the following derivation still applies but depends on the actual character of these fields, and is avoided for simplicity. An extensive account of such systems can be found in [23].

Having restricted H , H_i and F to be pure functionals of the canonical variables, the first terms in the identities (153) and (154) are compared. The evolution postulate implies that

$$\{H_j(x', t'), F(x'', t'')\} = \{H^D_j(x', t'), F(x'', t'')\}. \quad (155)$$

The use of the strong sign is due to the replacement of the field velocities, as already explained. Both brackets depend solely on the canonical variables because of the restrictions imposed. Therefore, a further application of the evolution postulate yields

$$\{\{H_j(x', t'), F(x'', t'')\}, H_i(x, t)\} = \{\{H^D_j(x', t'), F(x'', t'')\}, H^D_i(x, t)\}, \quad (156)$$

which is valid precisely because Hamilton's equations are preserved under the commutation with the Hamiltonian.

Repeating this argument when comparing the second terms in the identities (153) and (154), the following equation arises,

$$\{\{F(x'', t''), H_i(x, t)\}, H_j(x', t')\} = \{\{F(x'', t'')H^D_i(x, t)\}, H^D_j(x', t')\}. \quad (157)$$

Equations (156)-(157) implies that the remaining terms in the identities (153)-(154) should be equal,

$$\{\{H_i(x, t), H_j(x', t')\}, F(x'', t'')\} = \{\{H^D_i(x, t), H^D_j(x', t')\}, F(x'', t'')\} = 0. \quad (158)$$

The Poisson bracket between the two deformation generators in equation (158) is calculated to give the history analogue of the Dirac relation (14),

$$\{H^D_i(x, t), H^D_j(x', t')\} = H^D_j(x, t)\delta_i(x, x')\delta(t, t') - (ix \leftrightarrow jx'). \quad (159)$$

Then the evolution postulate is used once more to give an equation that holds exclusively on the canonical phase space,

$$\left\{ \left[\{H_i(x, t), H_j(x', t')\} - \left(H_j(x, t)\delta_i(x, x')\delta(t, t') - (ix \leftrightarrow jx') \right) \right], F(x'', t'') \right\} = 0. \quad (160)$$

Since it holds for any choice of the functional F, the following relation for the supermomenta arises,

$$\{H_i(x, t), H_j(x', t')\} = H_j(x, t)\delta_i(x, x')\delta(t, t') + C_{ij}[x, t; x', t'] - (ix \leftrightarrow jx'). \quad (161)$$

The term C_{ij} is just a constant function of its arguments.

The same argument can be applied to the mixed Jacobi identities

$$\begin{aligned} & \{ \{H_j(x', t'), F(x'', t'')\}, H(x, t) \} + \{ \{F(x'', t''), H(x, t)\}, H_j(x', t') \} \\ & + \{ \{H(x, t), H_j(x', t')\}, F(x'', t'') \} = 0, \end{aligned} \quad (162)$$

$$\begin{aligned} & \{ \{H^D_j(x', t'), F(x'', t'')\}, H^D(x, t) \} + \{ \{F(x'', t''), H^D(x, t)\}, H^D_j(x', t') \} \\ & + \{ \{H^D(x, t), H^D_j(x', t')\}, F(x'', t'') \} = 0, \end{aligned} \quad (163)$$

resulting in the relation

$$\{H(x, t), H_i(x', t')\} = H(x, t)\delta_i(x, x')\delta(t, t') + H_{,i}(x, t)\delta(x, x')\delta(t, t') + C_i[x, t; x', t'], \quad (164)$$

where C_i is constant.

● Recovery Of The Super-momentum Constraint: The situation changes considerably when the same argument is applied to the identities involving the super-Hamiltonians,

$$\begin{aligned} & \{\{H(x', t'), F(x'', t'')\}, H(x, t)\} + \{\{F(x'', t''), H(x, t)\}, H(x', t')\} \\ & + \{\{H(x, t), H(x', t')\}, F(x'', t'')\} = 0, \end{aligned} \quad (165)$$

$$\begin{aligned} & \{\{H^D(x', t'), F(x'', t'')\}, H^D(x, t)\} + \{\{F(x'', t''), H^D(x, t)\}, H^D(x', t')\} \\ & + \{\{H^D(x, t), H^D(x', t')\}, F(x'', t'')\} = 0. \end{aligned} \quad (166)$$

This leads to the relation

$$\{\{H(x, t), H(x', t')\}, F(x'', t'')\} = \{\{H^D(x, t), H^D(x', t')\}, F(x'', t'')\}, \quad (167)$$

whose left and right side is evaluated on the canonical and on the deformation phase space, respectively.

Considering the Poisson bracket between the deformation generators, the fact arises that the Dirac algebra is not a genuine Lie algebra but depends explicitly on the spatial metric,

$$\{H^D(x, t), H^D(x', t')\} = g^{ij}(x, t)H^D_i(x, t)\delta_{,j}(x, x')\delta(t, t') - (x \leftrightarrow x'). \quad (168)$$

Since the theory is by assumption independent of any fixed fields, it follows that the metric has to be a canonical variable in order to appear in equation (167).

The evolution postulate and the fact that the metric is a canonical variable can be used to write equation (167) exclusively in terms of variables defined on the canonical phase space[20],

$$\begin{aligned} & \left\{ \left[\{H(x, t), H(x', t')\} - \left(g^{ij}(x, t)H_i(x, t)\delta_{,j}(x, x')\delta(t, t') - (x \leftrightarrow x') \right) \right], F(x'', t'') \right\} \\ & = - \left(H_i(x, t)\delta_{,j}(x, x')\delta(t, t')\{g^{ij}(x, t), F(x'', t'')\} - (x \leftrightarrow x') \right). \end{aligned} \quad (169)$$

The term on the right side is the compensation needed in order that the metric be taken inside the Poisson brackets in the canonical phase space.

Because equation (169) is a linear first order equation that is required to hold for an arbitrary choice of functional F , it cannot be satisfied unless the super-momenta are constrained to vanish,

$$H_i(x, t) \simeq 0. \quad (170)$$

The proof is based on the following procedure. Both sides of equation (169) are expanded in terms of the spatial derivatives of the δ -functions. Then, because of the linearity and the specific form of the equation, particular choices of functionals F can be found that violate at least one of the terms in the expansion.

● Recovery Of The Weak Representation Postulate: The constraint (170) leads to

$$\left\{ \left[\{H(x, t), H(x', t')\} - \left(g^{ij}(x, t) H_i(x, t) \delta_{,j}(x, x') \delta(t, t') - (x \leftrightarrow x') \right) \right], F(x'', t'') \right\} \simeq 0, \quad (171)$$

which must still hold for every choice of functional F . Teitelboim argued[19] that the weak equation (171)—which in the equal-time approach is derived from the principle of path independence—is enough to imply that the expression

$$\{H(x, t), H(x', t')\} - \left(g^{ij}(x, t) H_i(x, t) \delta_{,j}(x, x') \delta(t, t') - (x \leftrightarrow x') \right) \quad (172)$$

vanishes strongly. Specifically, he argued that (172) must not depend on any canonical variables because, if it did, particular choices of functionals F could always be found to violate equation (171), in a process similar to the one described above. The quantity (172) should therefore be equal to a constant function, which is zero[19] because of the

requirement that the algebra be closed. The requirement of closure actually implies that the constant terms C_{ij} and C_i in equations (161) and (164) should also be zero[19] and, hence, the history analogue of the strong Dirac algebra is derived.

However, this argument is not true in general, because in a constrained system it must be ensured that all the terms in equation (171) are well-defined on the constraint surface. If any of the first partial derivatives of (172) does not vanish on the constraint surface, the argument in [19] can be applied indeed, and leads to the conclusion that the expression (172) is zero strongly. On the other hand, if both partial derivatives of (172) vanish weakly, well-defined choices for functionals F that violate equation (171) cannot be found, since this would require the first partial derivatives of any such F to have an infinite value on the constraint surface. Consequently, the most general expression for the algebra between the super-Hamiltonians is the weak Dirac relation mentioned in section 2,

$$\{H(x, t), H(x', t')\} = g^{ij}(x, t)H_i(x, t)\delta_{,j}(x, x')\delta(t, t') + G(x, t; x', t') - (x \leftrightarrow x'), \quad (173)$$

where both the first derivatives of G vanish on the constraint surface (170). Notice that any constant terms are absorbed in this definition of G .

The fact that the system is constrained in H_i demands for the re-examination of the assumptions that led to equations (161), (164) and (173). The only requirement for the validity of the previous procedure is the preservation of any weak equality under the commutation with the canonical generators. However, this is included already in the definition of the evolution postulate for constrained systems, so any strong equality signs must be replaced simply with weak ones.

This replacement results in the complete history analogue of the weak Dirac algebra (101-103) as well as in the weak re-shuffling and ultra-locality conditions and in the rest of the weak evolution postulate. Notice that, although the term “weak” refers currently to the constraint surface (170), the arguments used do not depend on the actual definition of the constraint surface. Therefore, the present conclusions will remain valid in case that the super-Hamiltonian is proved to be constrained.

- **The Principle Of Path Independence:** The path independence of the dynamical evolution does not have to be assumed separately in the present method. Instead, it is a consequence of the evolution postulate. This can be shown directly by starting from the evolution postulate and the derived weak Dirac algebra, and then repeating in the reverse order the procedure used in [19]. It follows immediately that the change in the canonical variables during the dynamical evolution of the theory is independent of the path used in its actual evaluation. An alternative proof uses the fact that the principle of path independence is a direct consequence of the integrability of Hamilton’s equations. The evolution postulate is just another name for these equations and, therefore, any solution of the postulate will lead automatically to a path-independent dynamical evolution.

- **Recovery Of The Super-Hamiltonian Constraint:** When the representation postulate is imposed in the weak sense, the super-Hamiltonian constraint does not follow immediately from the closure of the Dirac algebra, as in [20], but it is also necessary to take into account the actual form of equations (141-144). These equations are considered below in the case of general relativity or, more accurately, in the case when the configuration variable is the pullback of the spacetime metric.

Referring to the corresponding comment at the end of chapter III-3, the most general form of the weak evolution postulate is the following:

$$\begin{aligned} \{g_{ij}(x, t), H(x', t')\} &= 2n_{\alpha;\beta}(x, t)\mathcal{X}^\alpha{}_i(x, t)\mathcal{X}^\beta{}_j(x, t)\delta(x, x')\delta(t, t') \\ &+ V_{ij}(x, t; x', t'), \end{aligned} \quad (174)$$

$$\begin{aligned} \{g_{ij}(x, t), H_k(x', t')\} &= g_{ki}(x, t)\delta_{,j}(x, x')\delta(t, t') + g_{kj}(x, t)\delta_{,i}(x, x')\delta(t, t') \\ &+ g_{ij,k}(x, t)\delta(x, x')\delta(t, t') + V_{ijk}(x, t; x', t'), \end{aligned} \quad (175)$$

$$\{p^{ij}(x, t), H(x', t')\} = \{p^{ij}[\mathcal{X}(x, t)], H^D(x', t')\} + W^{ij}(x, t; x', t'), \quad (176)$$

$$\{p^{ij}(x, t), H_k(x', t')\} = \{p^{ij}[\mathcal{X}(x, t)], H^D{}_k(x', t')\} + W^{ij}{}_k(x, t; x', t'). \quad (177)$$

The tensors V_{ij} , V_{ijk} , W^{ij} and $W^{ij}{}_k$ depend on the canonical fields and are required to vanish on the constraint surface $H_i \simeq 0$. Because of the existence of the additional terms, the general solution of the coupled set (174-177) cannot be found explicitly. Nevertheless, the form of the evolution postulate allows some definite conclusions to be drawn, a part of which can be used to prove that the Hamiltonian is constrained.

The important observation[20] is that the conjugate momentum p^{ij} must be a tensor density of weight one in order that the form $p^{ij}\delta g_{ij}$ that appears in the canonical action be coordinate independent. As a result, the Poisson brackets between the tangential deformation generator and p^{ij} depend only on the weight of the latter, and equation (177) becomes

$$\begin{aligned} \{p^{ij}(x, t), H_k(x', t')\} &= \delta^j{}_k p^{im}(x, t)\delta_{,m}(x, x')\delta(t, t') + \delta^i{}_k p^{jm}(x, t)\delta_{,m}(x, x')\delta(t, t') \\ &- p^{ij}(x, t)\delta_{,k}(x, x')\delta(t, t') - p^{ij}{}_{,k}(x, t)\delta(x, x')\delta(t, t') \end{aligned}$$

$$+W^{ij}_k(x, t; x', t'). \quad (178)$$

Consider therefore a solution (H, H_i) of the system (174-177), taking into account equation (178). By the assumption of existence of such a solution, the left sides of equations (175) and (178) must satisfy the integrability condition

$$\{\{g_{ij}(x, t), H_k(x', t')\}, p^{mn}(x'', t'')\} = \{\{p^{mn}(x'', t''), H_k(x', t')\}, g_{ij}(x, t)\}. \quad (179)$$

Because the non-vanishing terms in equations (175) and (178) are integrable[20], the weakly vanishing terms in the same equations should also be integrable,

$$\{V_{ijk}(x, t; x', t'), p^{mn}(x'', t'')\} = \{W^{mn}_k(x'', t''; x', t'), g_{ij}(x, t)\}. \quad (180)$$

This implies that functionals H^*_i and K_i can be found, satisfying

$$\begin{aligned} \{g_{ij}(x, t), H^*_k(x', t')\} &= g_{ki}(x, t)\delta_{,j}(x, x')\delta(t, t') + g_{kj}(x, t)\delta_{,i}(x, x')\delta(t, t') \\ &+ g_{ij,k}(x, t)\delta(x, x')\delta(t, t'), \end{aligned} \quad (181)$$

$$\begin{aligned} \{p^{ij}(x, t), H^*_k(x', t')\} &= \delta^j_k p^{im}(x, t)\delta_{,m}(x, x')\delta(t, t') + \delta^i_k p^{jm}(x, t)\delta_{,m}(x, x')\delta(t, t') \\ &- p^{ij}(x, t)\delta_{,k}(x, x')\delta(t, t') - p^{ij}_k(x, t)\delta(x, x')\delta(t, t'), \end{aligned} \quad (182)$$

$$\{g_{ij}(x, t), K_k(x', t')\} = V_{ijk}(x, t; x', t'), \quad (183)$$

$$\{p^{ij}(x, t), K_k(x', t')\} = W^{ij}_k(x, t; x', t'). \quad (184)$$

It follows from equations (181-184) that every solution H_i of the weak evolution postulate can be written as the sum of two terms,

$$H_i = H^*_i + K_i. \quad (185)$$

Furthermore, the form of H^*_i is uniquely fixed by equations (181) and (182), and corresponds to the super-momentum of general relativity,

$$H^*_i = \mathcal{H}^{gr}_i, \quad (186)$$

written out in equation (11).

It can now be shown that the super-Hamiltonian of the theory is constrained. As in [20], this follows from the preservation of the super-momentum constraint under the dynamical evolution, resulting in the condition

$$\{H(x, t), H_i(x', t')\} \simeq 0. \quad (187)$$

Using equations (185) and (186), this condition can be written as

$$\{H(x, t), [\mathcal{H}^{gr}_i(x', t') + K_i(x', t')]\} \simeq 0 \quad (188)$$

or, equivalently, as

$$\{H(x, t), \mathcal{H}^{gr}_i(x', t')\} \simeq 0. \quad (189)$$

This follows from equations (183-184) and from the fact that W^{ij}_k and V_{ijk} vanish on the constraint surface (170).

The evolution postulate can be used once more to rewrite equation (189) as follows,

$$\{H(x, t), \mathcal{H}^{gr}_i(x', t')\} \simeq \{H(x, t), H^D_i(x', t')\} \simeq 0. \quad (190)$$

Notice that this is a special application of the evolution postulate where the arbitrary test functional F has been replaced by the super-Hamiltonian. The latter must transform

necessarily as a scalar density of weight one[20],

$$\{H(x, t), H^D_i(x', t')\} \simeq H(x, t)\delta_{,i}(x, x')\delta(t, t') + H_{,i}(x, t)\delta(x, x')\delta(t, t'), \quad (191)$$

so by combining equations (190) and (191) the constraint $H \simeq 0$ arises. Recall that the actual definition of the constraint surface does not affect the validity of any of the above arguments, and hence the procedure just described remains consistent under the additional constraint.

- The Representations Of The Weak Principle: Although the understanding of the relationship between the strong and the weak equations is no longer an issue (in the revised algorithm no strong equations are used) there is still need to clarify the relation between the “strong” and “weak” representations of the evolution postulate. In particular, there is need to understand exclusively in terms of the evolution postulate how the standard representation of general relativity arises and, also, to find out if the new representations are physically equivalent to the standard one. “Physically equivalent” means that they must generate weakly the same equations of motion and lead to the same constraint surface.

A preliminary examination of this problem has already been carried out when proving that the super-Hamiltonian of the theory is constrained. Indeed, equation (185) shows that the standard representation of the super-momentum is recovered from the evolution postulate as the special case $K_i = 0$. Also, equations (175) and (178) imply that all solutions H_i generate weakly the same equations of motion. Finally, it follows from equation (185) and from the fact that both partial derivatives of K_i vanish on the constraint surface $H_i = 0$ that the constraints H_i and \mathcal{H}^{gr}_i imply each other. The representations H_i and \mathcal{H}^{gr}_i are

therefore physically equivalent, and the privileged position occupied by \mathcal{H}^{gr}_i is merely because the standard description of the system is minimal.

On the other hand, whether the same is true for the representations of H cannot be said without further examination. The basic complication arises because equation (135) can only be inverted *implicitly* in order that the momenta be defined as functionals of the embedding. In addition, when the field velocities are replaced on the right side of equations (174) and (176), the resulting expressions are not the same for all representations. This spoils the method used when deriving the representations for the super-momentum.

The issue concerning the physical equivalence of the “weak” representations is therefore still unclear. It would be certainly interesting if representations could be found that are not equivalent to the standard super-Hamiltonian, but this possibility is rather remote considering the restrictions imposed on the spacetime character of any such representations by Lovelock’s theorem[31].

The question arises whether an additional postulate is missing, that could uniquely select the “strong” representation of general relativity. This is not difficult to be found, and simply corresponds to requiring all the weakly vanishing terms V_{ij} , V_{ijk} , W^{ij} and W^{ij}_k in equations (174)-(177) to be identically zero. This requirement indeed leads to the standard canonical representation derived by Kuchař *et al* in [20]. However, it cannot be justified by any *physical* principle because, at the very least, it is a general property of constrained systems to allow many different sets of equivalent constraints. Therefore, the best that can be achieved is an actual proof that all the “weak” representations are equivalent.

The conclusion that postulates (a) and (b) are the only assumptions needed in the deriva-

tion of the most general canonical representation is more transparent in the following chapter. There, the history algorithm is applied directly to a simple unconstrained system, namely the scalar field theory on a given metric background.

Chapter IV-2. The Direct Method.

Up to now, the intricate nature of general relativity has not allowed any details of the history formalism to be revealed. For this reason, a sufficiently simpler system is considered below. It is derived directly from the evolution postulate, thus providing a clear illustration of the new formalism.

● Preliminaries: A scalar field theory with a non-derivative coupling to the metric is considered. Because of the restriction on the coupling, the Lagrangian of the theory can be written in the following form,

$$\mathcal{L}(x, t) = \mathcal{L}\left[\phi, \dot{\phi}, N, g, N^i \phi_{,i}, g^{kj} \phi_{,k} \phi_{,j}\right](x, t). \quad (192)$$

To be precise, the Lagrangian may depend on the additional combinations

$$\begin{aligned}
& g^{ij} N_{,i} N_{,j} , \\
& g^{ij} \phi_{,i} N_{,j} , \\
& N^i N_{,i} , \\
& g_{ij} N^i N^j
\end{aligned} \tag{193}$$

which are also compatible with the assumption of the coupling. However, using similar arguments to the ones that follow, it can be shown that this dependence is trivial. For simplicity, it is taken to be trivial from the beginning.

The evolution postulate states that the following conditions should be satisfied,

$$\{\phi(x, t), H(x', t')\} \simeq \{\phi[\mathcal{X}](x, t), \mathcal{P}_\beta(x', t')\} n^\beta(x', t'), \tag{194}$$

$$\{\phi(x, t), H_i(x', t')\} \simeq \{\phi[\mathcal{X}](x, t), \mathcal{P}_\beta(x', t')\} \mathcal{X}^\beta_i(x', t'), \tag{195}$$

$$\{\pi(x, t), H(x', t')\} \simeq \{\pi[\mathcal{X}](x, t), \mathcal{P}_\beta(x', t')\} n^\beta(x', t'), \tag{196}$$

$$\{\pi(x, t), H_i(x', t')\} \simeq \{\pi[\mathcal{X}](x, t), \mathcal{P}_\beta(x', t')\} \mathcal{X}^\beta_i(x', t'), \tag{197}$$

The weak sign refers to the still unknown Hamilton's equations. Initially, the most general canonical representation for H and H_i is sought that satisfies the unconstrained version of the evolution postulate.

Recall that, according to the plan described in chapter III-3, any time derivatives of the canonical variables in equations (194)-(197) should be replaced with the corresponding expressions arising from Hamilton's equations,

$$\frac{\partial}{\partial t} \phi(x, t) \simeq \int d^3 x' dt' \{\phi(x, t), (NH + N^i H_i)(x', t')\}, \tag{198}$$

$$\frac{\partial}{\partial t}\pi(x, t) \simeq \int d^3x' dt' \{\pi(x, t), (NH + N^i H_i)(x', t')\}. \quad (199)$$

This replacement results in a system of coupled differential equations for the generators H and H_i , the general solution of which corresponds to the general canonical representation compatible with the postulate.

The above procedure is rather formal, so it is replaced below with one that is more suitable for the present purposes. Specifically, the super-Hamiltonian is not considered to be the “unknown” of the problem but, instead, the Legendre relation between the field velocity and the momentum takes its place. Because the information incorporated in the latter is not equally rich, the need that the Lagrangian be treated as a “new” unknown will arise at some stage.

● The Unknown Legendre: According to these modifications, the momentum is expressed as a functional of the embedding according to

$$\pi[\mathcal{X}](x, t) := \Pi\left[\phi, \dot{\phi}, N, g, N^i \phi_{,i}, g^{kj} \phi_{,k} \phi_{,j}\right](x, t). \quad (200)$$

The function Π is the required Legendre relation. Its form is the most general possible, assuming that the Lagrangian of the theory is given by equation (192). Notice that the variables treated as arguments of Π are independent of each other, since the system is unconstrained by assumption.

To proceed further, the following history Poisson brackets are needed:

$$\{\phi, \mathcal{P}_\beta'\} = \phi_{,\beta} \delta\delta, \quad (201)$$

$$\{\dot{\phi}, \mathcal{P}_\beta'\} = \phi_{,\beta} \delta\dot{\delta} + \phi_{,\beta\alpha} \dot{\mathcal{X}}^\alpha \delta\delta, \quad (202)$$

$$\{N, \mathcal{P}_\beta'\} = -n_\beta \delta \dot{\delta} + n_\beta N^m \delta_{,m} \delta - \frac{1}{2} N \gamma_{\mu\nu, \beta} n^\mu n^\nu \delta \delta, \quad (203)$$

$$\{g, \mathcal{P}_\beta'\} = 2g \mathcal{X}_\beta^m \delta_{,m} \delta + g \gamma_{\mu\nu, \beta} \mathcal{X}^\mu_m \mathcal{X}^{\nu m} \delta \delta, \quad (204)$$

$$\begin{aligned} \{N^i \phi_{,i}, \mathcal{P}_\beta'\} &= \mathcal{X}_\beta^i \phi_{,i} \delta \dot{\delta} + N n_\beta g^{im} \phi_{,i} \delta_{,m} \delta - N^m \mathcal{X}_\beta^i \phi_{,i} \delta_{,m} \delta + \\ &+ N \gamma_{\mu\nu, \beta} n^\mu \mathcal{X}^{\nu i} \phi_{,i} \delta \delta + \phi_{, \beta} N^i \delta_{,i} \delta + \phi_{, \beta \alpha} N^i \mathcal{X}^\alpha_{,i} \delta \delta, \end{aligned} \quad (205)$$

$$\begin{aligned} \{g^{ij} \phi_{,i} \phi_{,j}, \mathcal{P}_\beta'\} &= -\mathcal{X}_\beta^i g^{jm} \phi_{,i} \phi_{,j} \delta_{,m} \delta - \mathcal{X}_\beta^j g^{im} \phi_{,i} \phi_{,j} \delta_{,m} \delta + \\ &+ \phi_{, \beta} g^{ij} \phi_{,j} \delta_{,i} \delta + \phi_{, \beta \alpha} g^{ij} \phi_{,j} \mathcal{X}^\alpha_{,i} \delta \delta - \gamma_{\mu\nu, \beta} \mathcal{X}^{\mu i} \mathcal{X}^{\nu j} \phi_{,i} \phi_{,j} \delta \delta. \end{aligned} \quad (206)$$

In the above equations, $\delta(x, x')\delta(t, t')$, $\frac{\partial}{\partial x^i} \delta(x, x')\delta(t, t')$ and $\delta(x, x')\frac{\partial}{\partial t} \delta(t, t')$ are denoted, respectively, by $\delta\delta$, $\delta_{,i} \delta$ and $\delta \dot{\delta}$. If an expression is evaluated at (x', t') it is primed.

Starting from equations (196)-(197), the terms that are proportional to the time derivative of the delta function are considered,

$$\left\{ \pi[\mathcal{X}](x, t), \mathcal{P}_\beta'(x', t') \right\} \simeq \left[\frac{\partial \Pi}{\partial \dot{\phi}} \phi_{, \beta} - \frac{\partial \Pi}{\partial N} n_\beta + \frac{\partial \Pi}{\partial [N^m \phi_{,m}]} \phi_{,k} \mathcal{X}_\beta^k \right] \delta \dot{\delta} + \text{rest of terms}. \quad (207)$$

Since the left side of the above equation is independent of any time derivatives by construction, the terms multiplying the derivative of the delta function must vanish weakly. However, the system has been assumed to be unconstrained, so the only way that these terms can vanish is that they vanish strongly.

Taking the normal and tangential projections of equation (207), two differential equations are obtained,

$$\begin{aligned} \frac{\partial \Pi}{\partial \dot{\phi}} N^{-1} [\dot{\phi} - N^m \phi_{,m}] + \frac{\partial \Pi}{\partial N} &= 0 \\ \frac{\partial \Pi}{\partial [N^m \phi_{,m}]} + \frac{\partial \Pi}{\partial \dot{\phi}} &= 0. \end{aligned} \quad (208)$$

The most general Legendre relation Π that solves the above equations has the form

$$\pi[\mathcal{X}](x, t) = \Pi \left[N^{-1}[\dot{\phi} - N^i \phi_{,i}], \phi, g^{kj} \phi_{,k} \phi_{,j}, g \right] (x, t). \quad (209)$$

Next, equations (195) and (197) are compared. Because the field momentum must transform as a scalar density of weight one (appendix F), these equations fix the form of the super-momentum uniquely. It is given by

$$H_i(x, t) = \pi \phi_{,i}(x, t). \quad (210)$$

To be precise, a function $c_i(x, t)$ of space and time should also be added on the right side of the above equation, but it is set to zero because the Lagrangian is of the form (192).

Recall that the requirement concerning the weight of π has been the only assumption used in the derivation of equation (210). By imposing this requirement on the function Π as well, and by making use of the Poisson bracket relations (201)-(206), the following differential equation arises,

$$\Pi = [g^{\frac{1}{2}}] \frac{\partial \Pi}{\partial [g^{\frac{1}{2}}]}. \quad (211)$$

It admits the solution

$$\pi[\mathcal{X}](x, t) = [g^{\frac{1}{2}}] \Pi \left[N^{-1}[\dot{\phi} - N^i \phi_{,i}], g^{kj} \phi_{,k} \phi_{,j}, \phi \right] [\mathcal{X}(x, t)], \quad (212)$$

where Π transforms as a scalar density of weight zero. It can be shown that Π drops out of the remaining evolution postulate, which means that all the information incorporated in Π has been used. The final representation for the latter is given by equation (212).

● The Unknown Lagrangian: Returning to equations (196)-(197), the remaining terms are

considered. The relevant Poisson brackets are

$$\begin{aligned}
\left\{ \left[N^{-1}[\dot{\phi} - N^i \phi_{,i}] \right], \mathcal{P}_\beta \right\} n^\beta &= g^{km} \phi_{,m} \delta_{,k} \delta + \text{ultralocal terms}, \\
\left\{ g^{\frac{1}{2}}, \mathcal{P}_\beta' \right\} n^\beta &= 0 + \text{ultralocal terms}, \\
\left\{ [g^{mj} \phi_{,m} \phi_{,j}], \mathcal{P}_\beta' \right\} n^\beta &= 2 \left[N^{-1}[\dot{\phi} - N^i \phi_{,i}] \right] g^{km} \phi_{,m} \delta_{,k} \delta + \text{ultralocal terms}, \\
\left\{ \phi, \mathcal{P}_\beta' \right\} n^\beta &= 0 + \text{ultralocal terms},
\end{aligned} \tag{213}$$

and lead to the following equation,

$$\left\{ \pi, H' \right\} = g^{\frac{1}{2}} \left[\frac{\partial \Pi}{\partial [N^{-1}[\dot{\phi} - N^i \phi_{,i}]]} + 2N^{-1}[\dot{\phi} - N^i \phi_{,i}] \frac{\partial \Pi}{\partial [g^{kj} \phi_{,k} \phi_{,j}]} \right] g^{km} \phi_{,m} \delta_{,k} \delta. \tag{214}$$

The Lagrangian (192) is now treated as the new unknown of the problem. The super-Hamiltonian on the left side of equation (214) is expressed as a function of π and \mathcal{L} according to the usual Legendre definition

$$H := \frac{1}{N} \left[\left(\pi \dot{\phi} - \mathcal{L} \right) - N^i H_i \right] \tag{215}$$

and, similarly, the function Π appearing on the right side of (214) is replaced by the definition

$$\Pi := \frac{\partial}{\partial \dot{\phi}} \mathcal{L}. \tag{216}$$

Putting the definitions (215)-(216) in equation (214), the quantities H_i , $\dot{\phi}$, N and N^i drop out identically and a differential equation arises,

$$L_{RR} + 2RL_{RS} + 2L_S = 0. \tag{217}$$

As before, the subscripts indicate partial differentiation.

The function L is defined by

$$\mathcal{L} := N g^{\frac{1}{2}} L, \quad (218)$$

while the quantities R and S are defined by

$$R := N^{-1}[\dot{\phi} - N^i \phi_{,i}] \quad (219)$$

and

$$S := g^{kj} \phi_{,k} \phi_{,j}. \quad (220)$$

The general solution of (217) fixes the final form of \mathcal{L} ,

$$\mathcal{L} = N g^{\frac{1}{2}} L[(R^2 - S), \phi], \quad (221)$$

which can be recognized as the most general Lagrangian for a field theory with a non-derivative coupling to the metric[19]. In order that the derivation be complete, the ultra-local terms in equations (196)-(197) must be compared. These terms are cancelled identically and, therefore, overall consistency is reached.

PART V:

The Geometric Interpretation Of The New Algebra.

Chapter V-1. The Generators Of Deformations.

● Preliminaries: The method developed in [20] is used below as a means of finding the physical solutions of the genuine Lie algebra. As explained in chapters III-1 and IV-1, this is possible precisely because the ultra-locality and re-shuffling assumptions follow uniquely from the evolution postulate. The argument that leads to the required representations is based on the following observation.

If the canonical generators K , K_i are pure functionals of the canonical variables, they satisfy exactly the same algebra that is satisfied by the corresponding generators of hyper-surface deformations. It is only when prescribed fields are present in the theory that the canonical algebra may differ, according to the discussion in chapter IV-1. A representation for the deformation generators K^D , K_i^D arises when the deformation vector field is decomposed in an appropriate basis $(\mu^\alpha, \mu^\alpha_i)$,

$$K := \mu^\alpha[\mathcal{X}]P_\alpha, \quad (222)$$

$$K_i := \mu^\alpha_i[\mathcal{X}]P_\alpha, \quad (223)$$

so that the genuine Lie algebra is satisfied,

$$\{K(x, t), K(x', t')\} = 0, \quad (224)$$

$$\{K(x, t), K_i(x', t')\} = K_{,i}(x, t)\delta(x, x')\delta(t, t') + K(x, t)\delta_{,i}(x, x')\delta(t, t'), \quad (225)$$

$$\{K_i(x, t), K_j(x', t')\} = K_j(x, t)\delta_{,i}(x, x')\delta(t, t') - (ix \leftrightarrow jx'). \quad (226)$$

If the basis $(\mu^\alpha, \mu^\alpha_i)$ is used in the decomposition of the spacetime Lagrangian the canonical

generators are guaranteed to satisfy equations (224)-(226), provided that no “remnants” of μ^α or μ^α_i have been left in the theory in the form of prescribed functions. This is always the case for an unconstrained system, since any such remnants can be eliminated through parametrization. On the other hand, for a system that is constrained, the situation is more complicated and additional assumptions have to be made. These are discussed in part VI.

● The Representation For The Deformation Generators: The most general representation for K^D and K^D_i is required, treating the embedding \mathcal{X}^α and its conjugate momentum P_α as the sole variables. Here, there are no ultra-locality or re-shuffling requirements because the generators K^D and K^D_i are unknown themselves. However, there is still sufficient information to select the representation uniquely. The requirement that plays the role of the additional selection criteria is evidently the linearity in the momentum conjugate to the embedding.

The problem should not be made unnecessarily complicated, so the basis μ^α_i is identified with the usual coordinate basis \mathcal{X}^α_i . Accordingly, the identification of K^D_i with the generator of hyper-surface deformations H^D_i is implied. In this case, equation (226) is satisfied identically while the second relation (225) depends only on the transformation properties of K^D and, for the moment, can be ignored.

Using equations (222) and (223), the Poisson bracket in equation (224) is expanded according to

$$P_\alpha(x, t)\{\mu^\alpha(x, t), P_\beta(x', t')\}\mu^\beta(x', t') - (x \leftrightarrow x')(t \leftrightarrow t') = 0 \quad (227)$$

or, equivalently, according to

$$P_\alpha(x, t)\mu^\beta(x', t')\frac{\delta\mu^\alpha}{\delta\mathcal{X}^\beta}(x, t)\delta(x, x')\delta(t, t') - (x \leftrightarrow x')(t \leftrightarrow t') = 0. \quad (228)$$

The way in which the functional derivative of μ^α is expressed reflects the assumption that μ^α is a local functional of the embedding; that is, a function of \mathcal{X} and of a finite number of its spatial and time derivatives. Under this assumption, the functional derivative can be expanded according to

$$\frac{\delta\mu^\alpha}{\delta\mathcal{X}^\beta} := \frac{\partial\mu^\alpha}{\partial\mathcal{X}^\beta} + \frac{\partial\mu^\alpha}{\partial\mathcal{X}_i^\beta}\partial_i + \frac{\partial\mu^\alpha}{\partial\dot{\mathcal{X}}^\beta}\dot{\partial} + \frac{\partial\mu^\alpha}{\partial\dot{\mathcal{X}}_i^\beta}\dot{\partial}_i + \dots, \quad (229)$$

where the series has finite terms.

Next, the various terms in (228) are evaluated at (x, t) by using the series of identities

$$\begin{aligned} A(x', t')\delta\delta &= A\delta\delta, \\ A(x', t')\delta\dot{\delta} &= A\delta\dot{\delta} + \dot{A}\delta\delta, \\ A(x', t')\delta\ddot{\delta} &= A\delta\ddot{\delta} + 2\dot{A}\delta\dot{\delta} + \ddot{A}\delta\delta, \\ &e.t.c., \\ A(x', t')\delta_{,i}\delta &= A\delta_{,i}\delta + A_{,i}\delta\delta, \\ A(x', t')\delta_{,ij}\delta &= A\delta_{,ij}\delta + A_{,i}\delta_{,j}\delta + A_{,j}\delta_{,i}\delta + A_{,ij}\delta\delta, \\ &e.t.c., \\ A(x', t')\delta_{,i}\dot{\delta} &= A(x', t)\delta_{,i}\dot{\delta} + \dot{A}(x', t)\delta_{,i}\delta, \\ A(x', t')\delta_{,i}\dot{\delta} &= A(x, t')\delta_{,i}\dot{\delta} + A_{,i}(x, t')\delta\dot{\delta}, \\ A(x', t')\delta_{,i}\dot{\delta} &= A\delta_{,i}\dot{\delta} + \dot{A}\delta_{,i}\delta + A_{,i}\delta\dot{\delta} + \dot{A}_{,i}\delta\delta, \\ &e.t.c., \end{aligned} \quad (230)$$

and are then collected according to the corresponding derivatives of the δ -functions. The notation for the δ -functions is the same as the one used before.

Because of equation (229), each particular term has to vanish. This implies that the partial derivatives of μ^α with respect to any spatial or time derivatives of the embedding must be zero. Specifically, an iteration arises for the partial derivatives of μ^α , which then reduces to zero because the partial derivatives with respect to the highest-order spatial and time derivatives of the embedding have to be trivial. Crucial to the proof is the assumption that μ^α is a local functional of the embedding, so that highest-order spatial and time derivatives exist indeed.

Furthermore, because of the symmetry under the interchange of $K^D(x, t)$ with $K^D(x', t')$, the ultra-local terms in equation (229) vanish identically. As a result, no condition is imposed on the partial derivative of μ^α with respect to the embedding. The general solution for K^D then takes the form

$$K^D(x, t) = \mu^\alpha(\mathcal{X})(x, t)P_\alpha(x, t), \quad (231)$$

where $\mu^\alpha(\mathcal{X})$ is an ultra-local function of the embedding. This combination transforms as a scalar density of weight-one in equation (225) and provides the final solution to the representation problem.

The fact that μ^α is an ultra-local function of the embedding implies that $\mu^\alpha(\mathcal{X})$ corresponds to the pull-back of a spacetime vector field. This is the required geometric interpretation of the new Lie algebra.

Chapter V-2. The Canonical Generators.

● The Decomposition Of The Spacetime Theory: Having fixed the interpretation of the corresponding deformation generators, the evolution postulate can be used in exactly the same way as in parts III and IV. It determines the corresponding ultra-locality and re-shuffling conditions, the form of the canonical algebra and, finally, the actual form of the canonical representations. Equivalently, the spacetime action of the theory can be decomposed in terms of the new basis $(\mu^\alpha, \mathcal{X}^\alpha_i)$. This is a much simpler procedure, so it is the one that is followed below.

Suppose that the vector field μ^α is decomposed in the usual $(n^\alpha, \mathcal{X}^\alpha_i)$ basis according to

$$\mu^\alpha := An^\alpha + B^i \mathcal{X}^\alpha_i. \quad (232)$$

The functions A and B^i are identified as the normal and tangential projections of the vector field,

$$A = -n_\alpha \mu^\alpha, \quad (233)$$

$$B^i = \mathcal{X}_\alpha^i \mu^\alpha. \quad (234)$$

They transform as a spatial scalar and as a spatial vector respectively. Also suppose that the deformation vector of the foliation can be expanded in terms of the basis $(\mu^\alpha, \mathcal{X}^\alpha_i)$ according to

$$\dot{\mathcal{X}}^\alpha = M\mu^\alpha + M^i \mathcal{X}^\alpha_i. \quad (235)$$

The functions M and M^i can be viewed as the “new lapse” and the “new shift”.

The unconstrained canonical action (115) is then written in terms of A , B^i , M and M^i as follows,

$$\begin{aligned}
S[q^A, p_A] &= \int d^3x dt \left(p_A \dot{q}^A - \mathcal{H} \right), \\
\mathcal{H} &= MK + M^i K_i, \\
K &= AH + B^i H_i \\
K_i &= H_i.
\end{aligned} \tag{236}$$

The generators H and H_i are the usual generators arising from the decomposition of the action with respect to the lapse function and shift vector. The functions A and B^i are precisely the “remnants” of the vector field μ^α . This means that the generators K and H_i may not close according to the genuine algebra (224)-(224), and indeed they do not. As a result, the theory has to be parametrized.

● An Application: As an example, the action for a massless scalar field is considered. It is decomposed in terms of the new basis. The corresponding canonical generators are given by

$$K^\phi = \frac{\pi^2}{2g^{\frac{1}{2}}} + \frac{g^{\frac{1}{2}}}{2} g^{ij} \phi_{,i} \phi_{,j}, \tag{237}$$

$$\mathcal{H}_i^\phi = \pi \phi_{,i}. \tag{238}$$

When the theory is parametrized, the non-physical degrees of freedom $(\mathcal{X}^\alpha, P_\alpha)$ imply the constraints

$$K^T := K^\phi + \mu^\alpha P_\alpha \simeq 0, \tag{239}$$

$$H^T_i := \mathcal{H}_i^\phi + \mathcal{X}^\alpha_i P_\alpha \simeq 0. \tag{240}$$

Notice that the constraints have been projected along the appropriate basis $(\mu^\alpha, \mathcal{X}^\alpha_i)$.

Using the definitions (233)-(234), and the history Poisson brackets appearing in Appendix C, the required result arises:

$$\{K^T(x, t), K^T(x', t')\} = 0. \quad (241)$$

In addition, K transforms as a scalar density of weight one in equation (225), so the complete genuine Lie algebra is generated.

Part VI:

Discussion And Acknowledgments

Chapter VI-1. The Results, Their Relation, Their Extent And Limitations.

● The Genuine Algebra: A genuine Lie algebra was discovered by Brown and Kuchař in the context of gravity coupled to matter fields. A differential equation was constructed by Markopoulou that is satisfied by any scalar combinations of the gravitational constraints that close according to this algebra. Despite that, little was known about the significance of this equation either in gravity coupled to matter or in vacuum gravity. An insight into the origin of the algebra in the coupled theory was gained by constructing an action whose variation leads precisely to the general solution of Markopoulou’s equation.

However, this was not achieved via a canonical reduction, the physical meaning of which is transparent. Instead, the required algebraic manipulations were performed “outside” the canonical method, thus failing to connect the new algebra with the physical relevance of the theory. In particular, it remains unclear whether the new algebra can be maintained after the elimination of the scalar field momentum from the coupled action. Such an elimination is usually followed by a parametrization of the theory in terms of a privileged “time” associated with the (non-canonical) scalar field. If it can be shown that the new algebra is still present in the reduced action, a clear interpretation of the algebra in terms of “matter-time” will be achieved. This possibility was not explored here, and remains as a project for the future.

Priority was given instead to understanding the importance of the new algebra in vacuum

relativity. It was shown that all the gravitational combinations derived from the action functional generate a time evolution that is either zero or ill-defined on the constraint surface of the *vacuum* theory. As a result, alternative combinations were sought which not only satisfy the new algebra but also generate the appropriate dynamical evolution of Einstein's theory.

- The Use Of Classical Histories In The Thesis: For this purpose, an algorithm was generalized, originally used by Kuchař *et al* in the derivation of geometrodynamics from first principles. The need to adapt this algorithm to the requirements of the new algebra eventually led to the concept of classical histories. By using a Hamiltonian formalism defined over the space of histories, it was shown that the canonical representations of any theory can be derived from a minimal set of postulates. These depend only on the foliation through which the original spacetime theory is decomposed; therefore, they have a clear geometric interpretation.

This clarification—which is essential for the application of the algorithm to the new algebra—can only be achieved in the history formalism. This is simply because the new set of postulates contain derivatives of the canonical variables, and these cannot be defined in the standard approach. The implications for the quantization of general relativity are straightforward, and provide additional support for the consistent histories approach to quantum gravity[7,8,9,10].

A mistake in the original algorithm was also corrected. For unconstrained systems, both the old and new algorithms give identical results. For systems subject to constraints, the revised algorithm implies that certain strong equations (in the sense of Dirac) have to be

replaced by weak ones. As a result, new (weak) canonical representations of the evolution postulate arise.

The issue of the interpretation of the new algebra was finally discussed. The interpretation amounts to the decomposition of the deformation vector in terms of a particular foliation (other than the standard lapse-shift) so that the projections of the embedding momenta on the spacetime basis associated with the foliation close according to the given algebra. For the particular case of the new algebra the appropriate decomposition was shown to involve the projection on a spacetime vector field which is not normalized.

Having fixed the issue of the interpretation of the new algebra the generators of any canonical field theory are then made compatible with it. This is achieved by decomposing the spacetime Lagrangian with respect to the new spacetime basis and then parametrizing the result. The parametrization is essential in order that the canonical generators be pure functionals of the canonical variables. If this is not the case—i.e., if remnants of the spacetime basis are left in the theory—the procedure may not be effective.

- **Self-Commuting Combinations In The Vacuum Theory:** Unfortunately, general relativity is a constrained system, so the remnants arising from the vector field cannot be parametrized without changing the physical content of the theory. In the usual A.D.M. decomposition the corresponding remnants drop out of the canonical action because of the normalization conditions imposed on the normal vector field. Such a normalization cannot be used in the case of a spacetime vector field, which must be by definition independent of the embedding.

However, having located the root of the problem, it may be possible that this final dif-

ficuity can be overcome, possibly through the use of a covariant normalization condition on the new spacetime basis. If such a normalization can be attained the resulting gravitational constraints would be of particular importance, considering that the geometric interpretation of the new algebra is unique.

- **Histories And Explicit Spacetime Invariance:** An objection raised against canonical quantization of field theories in general, and Einstein's theory of spacetime in particular, is that the canonical formulation depends on a foliation and necessarily destroys the spacetime picture since it fails to keep track of the spacetime invariances of the action. In canonical quantum field theory, divergences are typically encountered when posing questions about probabilities at a single instant of time (a spacelike hypersurface). Such problems prompted previous attempts to establish the canonical structure in spacetime setting; either by using the Peirels bracket[32], or by imposing symplectic structure and inner product on the space of solutions[30], or by working in the multi-symplectic formalism[33].

The history symplectic structure is rather complicated for spacetime fields. Even for a scalar field on a given background the momentum conjugate to the spacetime field should be a spacetime scalar density, while the standard momentum depends on an auxiliary structure. The history symplectic structure that was used in the thesis depends on such a foliation, since the canonical fields are functions of the hypersurface coordinates (x^i, t) rather than of the original spacetime coordinates X^α . In order that the two pairs of coordinates be related, a foliation $X^\alpha = \mathcal{X}^\alpha(x^i, t)$ of spacetime into hypersurfaces is required. The possibility of constructing a *spacetime* history canonical formalism is currently investigated in collaboration with K. Kuchař[21]

● The Consistent Histories Approach: It is remarkable that most of the results described here could not have been achieved in the equal-time formulation without adding further structure. Whether the superiority of the history formalism can be established as a mathematical theorem depends on whether it is possible to construct a direct link between the equal-time and the history approaches. The clarification of this matter will certainly affect the way in which the consistent histories program is viewed. At the very least, there is the option of reformulating the complete canonical framework in terms of classical histories and searching for the classical analogues of the structures used in the consistent histories approach. At the very least, a classical insight into the idea of coarse-graining and into the related issue of probability is expected to be gained.

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Appendix A.

The equivalence between cases (46) and (47) is demonstrated; the same argument applies when comparing any two cases from the set (46)–(49). The sets of functions satisfying equations (46) and (47) are denoted respectively by (μ_1, λ_1, R_1) and (μ_2, λ_2, R_2) . For the two cases to be equivalent the following three conditions must be satisfied:

1. μ_1, λ_1, R_1 should satisfy (46)—the functions μ_1, λ_1 regarded as known—
2. μ_2, λ_2, R_2 should satisfy (47) and
3. $W[h, f] = \lambda_1^{\frac{\omega}{2}}(h - \mu_1 + R_1)^{\frac{\omega}{2}} = \lambda_2^{\frac{\omega}{2}}(h - \mu_2 + R_2)^{\frac{\omega}{2}}.$

To avoid comparing directly the differential equations arising from conditions 1 and 2 the following trick is performed. Equations (46) and (47) are inserted into equations (44) that determine the partial derivatives of $W[h, f]$. By condition 3, these equations must be the same for both cases. As a result, the differential equations (46) and (47) are transformed into the pair of algebraic equations

$$W_h = \frac{\lambda_1^{\frac{\omega}{2}}(h - \mu_1 + R_1)^{\frac{\omega}{2}}}{R_1} = -\frac{(\lambda_2)^{\frac{\omega}{2}}(h - \mu_2 + R_2)^{\frac{\omega}{2}}}{R_2} \quad (242)$$

$$W_f = \frac{\lambda_1^{\frac{\omega}{2}}(h - \mu_1 + R_1)^{\frac{\omega-2}{2}}}{R_1} = \frac{\lambda_2^{\frac{\omega}{2}}(h - \mu_2 + R_2)^{\frac{\omega-2}{2}}}{R_2}. \quad (243)$$

By comparing condition 3 with equations (242) and (243) the following *consistent* solution arises (notice that the system is over-determined):

$$\mu_2 = 2h - \mu_1 \quad \lambda_2^{\frac{\omega}{2}} = (-\lambda_1)^{\frac{\omega}{2}} \quad R_2 = -R_1. \quad (244)$$

This proves equivalence.

For completeness, the corresponding results arising from comparing cases (46) with (48) and (46) with (49) are written down:

$$\mu_3 = 2h - \mu_1, \quad (\lambda_3)^{\frac{\omega}{2}} = \frac{(-\lambda_1)^{\frac{\omega}{2}}(h - \mu_1 + R_1)}{(h - \mu_1 - R_1)}, \quad R_3 = R_1, \quad (245)$$

$$\mu_4 = \mu_1, \quad (\lambda_4)^{\frac{\omega}{2}} = \frac{(-\lambda_1)^{\frac{\omega}{2}}(h - \mu_1 + R_1)}{(h - \mu_1 - R_1)}, \quad R_4 = -R_1, \quad (246)$$

where (μ_3, λ_3) and (μ_4, λ_4) satisfy respectively equations (48) and (49).

Appendix B.

It is shown that the linear and the weight- ω equations are equivalent. The ansatz relation (42) can be considered as an one-parameter family of maps, a_ω , from the set of functions (μ, λ, R) satisfying equation (46) to the set of solutions W_ω of the corresponding weight- ω differential equation, on the assumption that W_ω , $W_{\omega H}$ and $W_{\omega F}$ are not identically zero

(see the remark at the beginning of this chapter). The reason that R is included in the set of functions (μ, λ, R) is that although R is a function of μ , defined by equation (43), it is not fully specified by μ due to the sign ambiguity.

One would like to know whether the map a_ω is one-to-one and, most importantly, whether it is onto. To check the latter, one supposes that W is any solution of the ω -equation (38), where—for simplicity—the subscript ω of W is omitted. A set of functions (μ, λ, R) obeying the linear equation (46) is therefore required, with the property of producing through the a_ω map the given solution W . This is similar to the procedure followed in Appendix A in order to show that equations (46)–(49) are equivalent; the difference is that the requirement that at least one of the cases (46)–(49) leads to W is lifted— W is now only confined to obey equation (38) for some weight ω .

The three conditions that must be satisfied are:

1. The original differential equation

$$\frac{\omega}{2}WW_f = fW_f^2 - \frac{1}{4}W_h^2, \quad W \neq 0 \quad W_f \neq 0 \quad W_h \neq 0. \quad (247)$$

2. The linear equation

$$\frac{1}{\lambda}\lambda_f - \frac{1}{R}\mu_f = 0 \quad \text{and} \quad \frac{1}{\lambda}\lambda_h - \frac{1}{R}\mu_h = 0; \quad R = \sqrt{(h - \mu)^2 - f}. \quad (248)$$

3. The ansatz relation

$$W = \lambda^{\frac{\omega}{2}} \left(h - \mu + \sqrt{(h - \mu)^2 - f} \right)^{\frac{\omega}{2}}. \quad (249)$$

The third condition can be written as

$$W^{\frac{2}{\omega}} = \lambda \left(h - \mu + \sqrt{(h - \mu)^2 - f} \right), \quad (250)$$

where this relation is valid up to a $\frac{2}{\omega}$ power of unity. Equation (250) can now be solved for μ , resulting in

$$\mu = h - \frac{1}{2} \left(\frac{W_{\omega}^{\frac{2}{\omega}}}{\lambda} + \frac{\lambda}{W_{\omega}^{\frac{2}{\omega}}} f \right). \quad (251)$$

Differentiating μ with respect to both h and f gives

$$\begin{aligned} \mu_h &= 1 - \frac{1}{\omega} \left(\frac{v^{\frac{2-\omega}{\omega}}}{\lambda} - \frac{\lambda}{v^{\frac{2+\omega}{\omega}}} f \right) W_h + \frac{1}{2} \left(\frac{W_{\omega}^{\frac{2}{\omega}}}{\lambda^2} - \frac{1}{v^{\frac{2}{\omega}}} f \right) \lambda_h \quad \text{and} \\ \mu_f &= -\frac{1}{2} \frac{\lambda}{W_{\omega}^{\frac{2}{\omega}}} - \frac{1}{\omega} \left(\frac{W_{\omega}^{\frac{2-\omega}}}{\lambda} - \frac{\lambda}{W_{\omega}^{\frac{2+\omega}}}} f \right) W_f + \frac{1}{2} \left(\frac{W_{\omega}^{\frac{2}{\omega}}}{\lambda^2} - \frac{1}{v^{\frac{2}{\omega}}} f \right) \lambda_f. \end{aligned} \quad (252)$$

Conditions 2 and 3—being now in the same form—are compared: In particular, substituting equation (251) into the expression for R —used in condition 2—one finds that

$$R = \frac{1}{2} \left(\frac{W_{\omega}^{\frac{2}{\omega}}}{\lambda} - \frac{\lambda}{v^{\frac{2}{\omega}}} f \right), \quad (253)$$

and hence condition 2 becomes

$$\mu_h = \frac{1}{2} \left(\frac{W_{\omega}^{\frac{2}{\omega}}}{\lambda^2} - \frac{1}{v^{\frac{2}{\omega}}} f \right) \lambda_h \quad \text{and} \quad \mu_f = \frac{1}{2} \left(\frac{W_{\omega}^{\frac{2}{\omega}}}{\lambda^2} - \frac{1}{v^{\frac{2}{\omega}}} f \right) \lambda_f. \quad (254)$$

When equations (254)—derived from the second condition—are compared with equations (252)—derived from the third condition—they lead to the following pair of equations for λ :

$$\omega - \left(\frac{v^{\frac{2-\omega}{\omega}}}{\lambda} - \frac{\lambda}{v^{\frac{2+\omega}{\omega}}} f \right) W_h = 0 \quad \text{and} \quad \frac{\lambda}{W_{\omega}^{\frac{2}{\omega}}} + \frac{2}{\omega} \left(\frac{W_{\omega}^{\frac{2-\omega}}}{\lambda} - \frac{\lambda}{W_{\omega}^{\frac{2+\omega}}}} f \right) W_f = 0. \quad (255)$$

The above set of equations admits a common solution for λ , we call it $\bar{\lambda}$, given by

$$\bar{\lambda} = -\frac{2W_{\omega}^{\frac{2}{\omega}} W_f}{W_h}. \quad (256)$$

Equations (255) and (50) are all well defined since—by assumption— W , W_h , and W_f are not identically zero, but for the two equations in (255) to be consistent, they must lead either to an identity or at least to a valid equation when $\bar{\lambda}$ is substituted into them. Indeed, by doing so, they both reduce to

$$\frac{\omega}{2}WW_f = fW_f^2 - \frac{1}{4}W_h^2, \quad (257)$$

which is of course true by virtue of condition 1. This proves that the a_ω map (42) (from the set of functions (μ, λ, R) to the set of solutions of the corresponding ω -equation (38)) is *onto*.

The expression for $\bar{\lambda}$ is now substituted back into equation (251) to give the relevant expression for μ ,

$$\bar{\mu} = h + \frac{W_f}{W_h}f + \frac{1}{4}\frac{W_h}{W_f}, \quad (258)$$

and the one for R ,

$$\bar{R} = \frac{W_f}{W_h}f - \frac{1}{4}\frac{W_h}{W_f}. \quad (259)$$

Note that the expression for \bar{R} is now sign-unambiguous.

To check if the map a_ω is one-to-one, two sets of functions (μ_1, λ_1, R_1) and (μ_2, λ_2, R_2) are considered. They are required to satisfy the linear pair of equations (46) and provide—through the a_ω map—the same solution W of the original ω -equation. The problem is already solved in Appendix A and leads to the following three conditions,

$$\begin{aligned} \lambda_1^{\frac{\omega}{2}}(h - \mu_1 + R_1)^{\frac{\omega}{2}} &= \lambda_2^{\frac{\omega}{2}}(h - \mu_2 + R_2)^{\frac{\omega}{2}}, \\ \frac{\lambda_1^{\frac{\omega}{2}}}{R_1}(h - \mu_1 + R_1)^{\frac{\omega}{2}} &= \frac{\lambda_2^{\frac{\omega}{2}}}{R_2}(h - \mu_2 + R_2)^{\frac{\omega}{2}}, \end{aligned}$$

$$\frac{\lambda_1^{\frac{\omega}{2}}}{R_1}(h - \mu_1 + R_1)^{\frac{\omega-2}{2}} = \frac{\lambda_2^{\frac{\omega}{2}}}{R_2}(h - \mu_2 + R_2)^{\frac{\omega-2}{2}}, \quad (260)$$

which admit the almost trivial solution

$$\mu_1 = \mu_2 \quad \lambda_1^{\frac{\omega}{2}} = \lambda_2^{\frac{\omega}{2}} \quad R_1 = R_2. \quad (261)$$

The word “almost” is used because of the ambiguity in the expression for the λ 's. However, if the equivalence class of λ is defined as the set of functions that differ from λ by an $\frac{\omega}{2}$ power of unity (it can be easily shown that this defines an equivalence relation) then each ansatz-map (42) becomes *one-to-one*. Hence $\bar{\mu}$, \bar{R} and the equivalence class of $(\bar{\lambda})$ —given respectively by equations (258), (101) and (256)—are unique. This completes the proof.

Appendix C.

In the following we denote $\delta(x, x')\delta(t, t')$ by $\delta\delta$, $\frac{\partial}{\partial x^i}\delta(x, x')\delta(t, t')$ by $\delta_{,i} \delta$ and $\frac{\partial}{\partial t}\delta(x, x')\delta(t, t')$ by $\delta \dot{\delta}$. If some expressions are calculated at (x', t') they will be simply primed.

$$\{\mathcal{X}^\alpha, \mathcal{P}_\beta\} = \delta^\alpha{}_\beta \delta \delta$$

$$\{\gamma_{\alpha\epsilon}, \mathcal{P}_\beta\} = \gamma_{\alpha\epsilon, \beta} \delta \delta$$

$$\{\gamma^{\alpha\epsilon}, \mathcal{P}_\beta\} = \gamma^{\alpha\epsilon}{}_{, \beta} \delta \delta$$

$$\{\delta^\alpha{}_\epsilon, \mathcal{P}_\beta\} = 0$$

$$\{\mathcal{X}^\alpha{}_i, \mathcal{P}_\beta\} = \delta^\alpha{}_\beta \delta_{,i} \delta$$

$$\{\mathcal{X}_{\alpha i}, \mathcal{P}_\beta\} = \gamma_{\alpha\beta} \delta_{,i} \delta + \gamma_{\alpha\mu, \beta} \mathcal{X}^\mu{}_i \delta \delta$$

$$\{\dot{\mathcal{X}}^\alpha, \mathcal{P}_\beta\} = \delta^\alpha{}_\beta \delta \dot{\delta}$$

$$\{n^\alpha, \mathcal{P}_\beta\} = -n_\beta \mathcal{X}^{\alpha m} \delta_{,m} \delta - \frac{1}{2} \gamma_{\mu\nu, \beta} n^\mu n^\nu n^\alpha \delta \delta - \gamma_{\mu\nu, \beta} n^\mu \gamma^{\alpha\nu} \delta \delta$$

$$\{n_\alpha, \mathcal{P}_\beta\} = -n_\beta \mathcal{X}_\alpha{}^m \delta_{,m} \delta - \frac{1}{2} \gamma_{\mu\nu, \beta} n^\mu n^\nu n_\alpha \delta \delta$$

$$\{g_{ij}, \mathcal{P}_\beta\} = \mathcal{X}_{\beta i} \delta_{,j} \delta + \mathcal{X}_{\beta j} \delta_{,i} \delta + \gamma_{\mu\nu, \beta} \mathcal{X}^\mu{}_i \mathcal{X}^\nu{}_j \delta \delta$$

$$\{g^{ij}, \mathcal{P}_\beta\} = -\mathcal{X}_\beta{}^i g^{jm} \delta_{,m} \delta - \mathcal{X}_\beta{}^j g^{im} \delta_{,m} \delta - \gamma_{\mu\nu, \beta} \mathcal{X}^{\mu i} \mathcal{X}^{\nu j} \delta \delta$$

$$\{\delta^i{}_j, \mathcal{P}_\beta\} = 0$$

$$\{\mathcal{X}^{\alpha i}, \mathcal{P}_\beta\} = -n^\alpha n_\beta g^{im} \delta_{,m} \delta - \mathcal{X}^{\alpha m} \mathcal{X}_\beta{}^i \delta_{,m} \delta - \gamma_{\mu\nu, \beta} \mathcal{X}^\alpha{}_m \mathcal{X}^{\nu m} \mathcal{X}^{\mu i} \delta \delta$$

$$\{\mathcal{X}_\alpha{}^i, \mathcal{P}_\beta\} = -n_\alpha n_\beta g^{im} \delta_{,m} \delta - \mathcal{X}_\alpha{}^m \mathcal{X}_\beta{}^i \delta_{,m} \delta - \gamma_{\mu\nu, \beta} n_\alpha n^\nu \mathcal{X}^{\mu i} \delta \delta$$

$$\{g, \mathcal{P}_\beta\} = 2g \mathcal{X}_\beta{}^m \delta_{,m} \delta + g \gamma_{\mu\nu, \beta} \mathcal{X}^\mu{}_m \mathcal{X}^{\nu m} \delta \delta$$

$$\{N, \mathcal{P}_\beta\} = -n_\beta \delta \dot{\delta} + n_\beta N^m \delta_{,m} \delta - \frac{1}{2} N \gamma_{\mu\nu, \beta} n^\mu n^\nu \delta \delta$$

$$\{N^i, \mathcal{P}_\beta\} = \mathcal{X}_\beta{}^i \delta \dot{\delta} + N n_\beta g^{im} \delta_{,m} \delta - N^m \mathcal{X}_\beta{}^i \delta_{,m} \delta + N \gamma_{\mu\nu, \beta} n^\mu \mathcal{X}^{\nu i} \delta \delta$$

(262)

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